Solutions for Matrix Notation Practice

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1. *Matrix operations:* For the matrices below, find $A - B$, AC, and $B'A$.

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 8 & 1 \\ 5 & 4 & 0 \end{bmatrix}$$

   Solution:

$$A - B = \begin{bmatrix} 1-1 & 4-3 \\ 2-1 & 6-4 \\ 3-2 & 8-5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1*3+4*5 & 1*8+4*4 & 1*1+4*0 \\ 2*3+6*5 & 2*8+6*4 & 2*1+6*0 \\ 3*3+8*5 & 3*8+8*4 & 3*1+8*0 \end{bmatrix} = \begin{bmatrix} 23 & 24 & 1 \\ 36 & 40 & 2 \\ 49 & 56 & 3 \end{bmatrix}$$

$$B' = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$$

$$B'A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 1+2+6 & 4+6+16 \\ 3+8+15 & 12+24+40 \end{bmatrix} = \begin{bmatrix} 9 & 26 \\ 26 & 76 \end{bmatrix}$$

2. *Matrix characteristics: Linear dependence.*

   (a) Are the columns of A (below) linearly dependent? Justify your answer.

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 1 & 0 & 5 \\ 1 & 0 & 5 \end{bmatrix}$$

   Solution:

$$C_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad C_2 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} \qquad C_3 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}$$

   $5C_1 - C_2 - C_3 = 0$, so $C_3 = 5C_1 - C_2$. Since C_3 can be expressed as a function of the other two vectors, C_1, C_2, and C_3 are not independent of each other; the columns of A are linearly dependent.

   Another solution (by row reduction):

$$\begin{vmatrix} 1 & 5 & 0 \\ 1 & 0 & 5 \\ 1 & 0 & 5 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 5 & 0 \\ 1 & 0 & 5 \\ 0 & 0 & 0 \end{vmatrix} \quad \text{with two non-zero rows remaining}$$

   Since $rank(A) = 2 < size(A) = 3$, the columns of A are linearly dependent.

(b) Is A an invertible matrix? Justify your answer.

Solution:

A is not an invertible matrix. An invertible matrix cannot have linear dependence (equivalently, it must be a full rank matrix). From the answer to part (a), we know that there is linear dependence (and that $rank(A) = 2 < size(A) = 3$ so A is less than full rank). Therefore A is not invertible.

(c) Verify that C (below) is the inverse of B (below).

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ -5 & -3 & -4 \end{bmatrix} \qquad C = \begin{bmatrix} -2 & -1 & -1 \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution:

$$
\begin{aligned}
BC &= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ -5 & -3 & -4 \end{bmatrix} \begin{bmatrix} -2 & -1 & -1 \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -2+2+1 & -1-1+2 & -1+0+1 \\ -4+2+2 & -2-1+4 & -2+0+2 \\ 10-6-4 & 5+3-8 & 5-0-4 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= I_3
\end{aligned}
$$

$$
\begin{aligned}
CB &= \begin{bmatrix} -2 & -1 & -1 \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ -5 & -3 & -4 \end{bmatrix} \\
&= \begin{bmatrix} -2-2+5 & -2-1+3 & -2-2+4 \\ 2-2+0 & 2-1+0 & 2-2+0 \\ 1+4-5 & 1+2-3 & 1+4-4 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= I_3
\end{aligned}
$$

BC = CB = I. Therefore C is the inverse of B.

3. *Regression models in matrix notation:* An observational study is being done in youth smokers. Y_i is each youth's average number of cigarettes smoked per day. The first covarate being considered is the youth's age in years (so X_{i1} is the age). The second covariate being considered is whether or not the youth participates in after-school youth groups, such as athletic teams or community service groups (so X_{i2} is 1 for yes and 0 for no). Consider the following regression equations for the first 6 youth in the study $(i = 1, 2, 3, 4, 5, 6)$:

$$
\begin{aligned}
Y_1 &= \beta_0 + \beta_1 * 13 & &+ \epsilon_1 \\
Y_2 &= \beta_0 + \beta_1 * 12 + \beta_2 &+ \epsilon_2 \\
Y_3 &= \beta_0 + \beta_1 * 16 & &+ \epsilon_3 \\
Y_4 &= \beta_0 + \beta_1 * 15 + \beta_2 &+ \epsilon_4 \\
Y_5 &= \beta_0 + \beta_1 * 12 & &+ \epsilon_5 \\
Y_6 &= \beta_0 + \beta_1 * 11 + \beta_2 &+ \epsilon_6
\end{aligned}
$$

Re-write these equations as one matrix equation. Clearly define each matrix in your equation.

Solution:

$$Y = X\beta + \epsilon$$

$$
Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}
\qquad
\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}
\qquad
\epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix}
$$

$$
X = \begin{bmatrix}
1 & x_{11} & x_{12} \\
1 & x_{21} & x_{22} \\
1 & x_{31} & x_{32} \\
1 & x_{41} & x_{42} \\
1 & x_{51} & x_{52} \\
1 & x_{61} & x_{62}
\end{bmatrix}
=
\begin{bmatrix}
1 & 13 & 0 \\
1 & 12 & 1 \\
1 & 16 & 0 \\
1 & 15 & 1 \\
1 & 12 & 0 \\
1 & 11 & 1
\end{bmatrix}
$$

4. *Quadratic forms:* Suppose we determine that participation in youth groups is not a significant predictor of cigarette smoking, so the covariate X_{i2} is dropped from the model. Now we just have one covariate X_{i1} for age. Least squares (or maximum likelihood) estimation of a simple linear regression $Y_i = \beta_0 + \beta_1 X_{i1} + \epsilon_i$ is done by finding those $\hat{\beta}_0$ and $\hat{\beta}_1$ which minimize

$$\sum_{i=1}^{m} (Y_i - \beta_0 - \beta_1 X_{i1})^2$$

(equivalently, $\sum_{i=1}^{m} \left(Y_i - \hat{Y}_i \right)^2$) where m is the total number of youth in the study. Re-write this summation in matrix notation.

Solution:

$Y_i = \beta_0 + \beta_1 x_{i1} + \epsilon_i, i = 1, \cdots, n$ is equivalent to $Y = X\beta + \epsilon$.

$\sum_{i=1}^{m} (Y_i - (\beta_0 + \beta_1 X_{i1}))^2$ is equivalent to $(Y - X\beta)' (Y - X\beta)$.