Proposition: All states in an irreducible Markov chain are of the same type: all recurrent or all transient and all have the same period.

Lemmas: Let $N_j \leq \infty$ denote the number of visits to state $j$ (after time 0) in a single infinite realization of the chain; i.e.,

$$N_j = \sum_{n=1}^{\infty} I\{X_n = j\}$$

Then,

(a) $P_j (N_j > n \mid X_0 = i) = f_{ij} f_{jj}^{n-1}, \quad n = 1, 2, \ldots$

(b) $E(N_j \mid X_0 = i) = \frac{f_{ij}}{1 - f_{jj}} = \begin{cases} \infty & \text{if } f_{ij} > 0, \\ f_{jj} = 1 & \text{if } f_{ij} = 0. \end{cases}$

Conj.

(f) $f_{jj} = 1 \iff \sum_{n=1}^{\infty} p_{jj}^{(n)} = \infty$

(g) $f_{jj} < 1 \iff \sum p_{ij}^{(n)} < \infty$.  

Proof: Note, $P_j (N_j = n \mid X_0 = i) = f_{ij} f_{jj}^{n-1} (1 - f_{jj})$.

If $f_{ij} = 1$ and $f_{jj} = 1$, then $j$ occurs infinitely often.
(b) \[ E(N_j \mid x_0 = i) = \sum_{n=1}^{\infty} R(N_j \geq n \mid x_0 = i) \]
\[ = \sum_{n=1}^{\infty} f_{ij} f_j^{(n-1)} = \frac{f_{ij}}{1 - f_{ij}} \]

(c) \[ E(N_j \mid x_0 = j) = \sum_{n=1}^{\infty} E(1 \mid x_n = j \mid x_0 = j) \]
\[ = \sum_{n=1}^{\infty} p_{ij}^{(n)} = \frac{f_{ij}}{1 - f_{ij}} = \infty \quad \text{iff} \quad f_{ij} = 1. \]

Note that \[ g_j(2) = G_j + F_j(2) \cdot G_j(2), \]
\[ \Rightarrow \quad f_{ij} = F_j(1), \quad G_j(1) = 1 + \sum_{n=1}^{\infty} p_{ij}^{(n)} \]

(d) follows from (c)

Proof of proposition: Periodicity and transience are class properties.
Theorem: Periodicity is an equivalence class property. i.e. if \( i \leftrightarrow j \), then \( d(i) = d(j) \).

Proof: Let \( p_{ij}^{(m)} > 0 \), \( p_{ji}^{(m)} > 0 \) and assume \( p_{ii}^{(m)} > 0 \). Then
\[
p_{ij}^{(m+n)} \geq p_{ij}^{(m)} p_{ij}^{(n)} > 0.
\]

Also,
\[
p_{ij}^{(m+n+s)} \geq p_{ij}^{(m)} p_{ij}^{(n)} p_{ij}^{(s)} > 0.
\]

So, \( d(i) \) must divide \( (m+n) \) and \( (m+n+s) \). Hence, it must divide the difference \( (m+n+s) - (m+n) = s \) for any \( s \) s.t. \( p_{ii}^{(s)} > 0 \).

Therefore, \( d(i) \) divides \( d(i) \). Similarly, \( d(i) \) divides \( d(i) \), so the two numbers must be equal.
Proposition: If state $i$ is recurrent, and state $i$ communicates with state $j$, then $j$ is recurrent.

Proof: Since state $i$ communicates with state $j$, there exist integers $k$ and $m$ such that $p_{ij}^{(k)} > 0$, $p_{ji}^{(m)} > 0$. Now, for any integer $n$,

$$p_{ij}^{(m+n+k)} > p_{ij}^{(m)} p_{ji}^{(n)} p_{ji}^{(k)}$$

Hence,

$$\sum_{n=1}^{\infty} p_{ij}^{(m+n+k)} > p_{ij}^{(m)} \sum_{n=1}^{\infty} p_{ji}^{(n)} p_{ji}^{(k)} = \infty$$

Similarly, transience is a class property.

The above proposition along with the fact that not all states in a finite MC can be transient leads to the fact that all states in a finite irreducible (all states communicate with each other) MC are recurrent. Similarly, positive recurrence is a class property. It can be shown that in a finite MC, all recurrent states are positive recurrent.
Proposition

If \( i \neq j \) are states of an irreducible aperiodic chain, then there is an integer \( N = N(i, j) \) such that \( p_{ij}^{(n)} > 0 \) for all \( n > N \).

proof: Since \( d(j) = 1 \), there are integers \( n_1, n_2 \) with \( \text{gcd}(1, n_1, n_2) = 1 \) such that \( p_{ij}^{(n_1)} > 0, n_2 = 1, 2 \).

Any sufficiently large \( n \) can be written as \( n = n_1 + kn_2 \), for non-negative integers \( k \) and \( n_2 \), where

\[
p_{ij}^{(n)} = p_{ij}^{(n_2)} > 0 \quad \text{for each } i, j.
\]

Also, for each pair \( i, j \), there is a \( n_0 \), s.t.

\[
p_{ij}^{(n_0)} > 0. \quad \text{Therefore,}
\]

\[
p_{ij}^{(n_0)} > 0, \quad \text{for all } n > n_0.
\]

Corollary: Let \( X \) and \( Y \) be iid irreducible aperiodic Markov chains. Then \( Z = (X, Y) \) is an irreducible Markov chain.

proof: \( p_{ij}^{x_t} = P \left[ z_{t+1} = (x, y) \mid z_t = (i, j) \right] \)

\[
= P \left( x_{t+1} = k, y_{t+1} = l \mid x_t = i, y_t = j \right)
= \frac{P(x_{t+1} = k, x_{t+1} = i)}{P(x_t = i)} \cdot \frac{P(y_{t+1} = l, y_{t+1} = j)}{P(y_t = j)} = p_{ik} \cdot p_{jl}
\]
A set $C$ of states is "closed" if $f_{jk} = 0$ for $j \in C$, $k \notin C$. Then

$$\sum p^{(n)}_{jk} = 0,$$

so we must have $p^{(n)}_{jk} = 0 \forall n$.

In fact, in order to verify that a set of states is closed we need to show that $p^{(2)}_{jk} = 0$ for $j \in C$, $k \notin C$, since

$$p^{(2)}_{jk} = \sum_{s \in S} p_{js} p_{sk} = \sum_{s \in S} p_{js} p_{sk} = 0.$$ 

We can use general $n$ follow by induction. If

$$x \leftrightarrow y \Rightarrow z, y \in C.$$

**Theorem:** If $S_T = \{ \text{transient states} \}$,

$$S_P = \{ \text{persistent states} \}$$

then

$$S = S_T + S_P$$

and $S_P = \bigcup_i C_i$ of disjoint irreducible closed sets.

**Proof:** Let $x \in S_P$ and define $C = \{ y \in S_P : x \rightarrow y \}$ by persistent $f_{xx} = 1$, so $x \in C$. Now $C$ is closed.

Because if $y \in C$ and $y \rightarrow z$, $z$ must be persistent and $x \rightarrow y \rightarrow z$, so $z \in C$.

Next we have to show that $C$ is irreducible.

Let $y \in C$, $z \in C$. We need to show that $y \rightarrow z$. 
ce $x \to y$, $y \to z$ by persistence. But $x \to z$

definition if $C$, so $y \to x \to z$.

The same argument with $y$ and $z$ transposed is that $z \to y$.

Let $C$ and $D$ be irreducible closed sets of $S$ and let $x \in C \cap D$.

Since $y \in C$. Since $C$ is irreducible, $y \notin C$. Since $D$ is closed, $x \in D$. If $x \to y$,

hence $y \in D$. Thus $C \subseteq D$. Similarly $D \subseteq C$

they are equal.