1. A machine may be in any one of the three states, labelled 1, 2, 3. If it is in state 1, it is in good shape and does not require any attention. If in state 2, a simple repair can be carried out to return the machine to state 2 or it can be left as it is. If it is in state 3, replacement is required.

Suppose that in the absence of any attention, the daily states of the machine follow a Markov chain with

$$P = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

Find the corresponding n-step transition matrix $P^n$. What is the distribution of the time taken to first reach state 3, starting from state 1. Suppose that the machine is checked every third day and that an immediate repair is carried out whenever the machine is found in state 2. Obtain the distribution of the time $T^* = 3, 6, \ldots$, until the machine is first found in state 3, starting from 1.

2. (a) Let $\{X_k\}$ be a Markov chain with t.p.m

$$P = \begin{pmatrix} .1 & .2 & .3 & .3 & .1 \\ .5 & .1 & .1 & .1 & .2 \\ .1 & .5 & .1 & .1 & .2 \\ .1 & .2 & .5 & .1 & .1 \\ .2 & .2 & .1 & .4 & .1 \end{pmatrix}$$

starting from 1, find the expected hitting time of state 5.
(b) Suppose that you do not know $P$, but have a realization $(x_k, k \leq 100)$ of $X_k$. Find a way to estimate the quantity in (a). Perform the estimation on a simulated realization.

3. A common assumption in Sociology that the social classes of successive generations in a family follow a Markov chain. Thus the occupation of a son is assumed to depend only on his father’s occupation and not on his grandfather’s. Suppose that the model is appropriate, then the t.p.m is given by

\[
\begin{pmatrix}
0.4 & 0.5 & 0.1 \\
0.05 & 0.7 & 0.25 \\
0.05 & 0.5 & 0.45
\end{pmatrix}
\]

Here the social classes are numbered 1 to 3, with 1 as the highest. Father’s class are the rows, son’s class the columns. A sociologist got a data from his colleague to illustrate the model. However it does not say whether the counts had father’s class along the rows or columns. The counts are:

\[
\begin{pmatrix}
7 & 6 & 5 \\
9 & 207 & 60 \\
2 & 64 & 60
\end{pmatrix}
\]

Propose a test that could be used to determine to more likely labeling. What is your conclusion?

4. Suppose that $p \in [0, 1)$ and that conditional on $p$, $X_1$ and $X_2$ are iid binary rv with $\Pr(X_i = 1|p) = p$ and $\Pr(X_i = 0|p) = 1 - p$, for $i = 1, 2$. Now suppose that $p$ is drawn from a uniform distribution on $[0, 1)$. Show that the correlation coefficient between $X_1$ and $X_2$ is $\frac{1}{3}$. 

2