Bayesian Sample Size Computations

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- Suppose we want to take a sample $y_1, \ldots, y_n \overset{iid}{\sim} N(\theta, \sigma^2)$. So $\bar{y} \sim N(\theta, \frac{\sigma^2}{n})$. Assume σ^2 is known. n is not known.

- Consider the classical hypothesis testing problem: $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta = \theta_1 > \theta_0$.

- Decision rule: Reject H_0 if $\bar{y} > \theta_0 + \frac{\sigma}{\sqrt{n}} z_{1-\alpha}$, where $\Phi(z_\alpha) = \alpha$ and $\Phi(\cdot)$ is the standard normal cdf.

- Requiring the procedure to have a power of at least $1-\beta$, we have:

$$
\begin{aligned}
1-\beta \leq P(Rej\ H_0 \,|\, H_1) &= P\left(\bar{y} > \theta_0 + \frac{\sigma}{\sqrt{n}} z_{1-\alpha} \,|\, \theta = \theta_1\right) \\
&= P\left(\frac{\sqrt{n}}{\sigma}(\bar{y}-\theta_1) > \frac{\sqrt{n}}{\sigma}(\theta_0-\theta_1) + z_{1-\alpha}\right) \\
&= P\left(Z > -\sqrt{n}\frac{\Delta}{\sigma} + z_{1-\alpha}\right) \\
&= 1-\Phi\left(-\sqrt{n}\frac{\Delta}{\sigma} + z_{1-\alpha}\right) \\
&= \Phi\left(\sqrt{n}\frac{\Delta}{\sigma} - z_{1-\alpha}\right) = \Phi\left(\sqrt{n}\frac{\Delta}{\sigma} + z_{\alpha}\right),
\end{aligned}
$$

where $\Delta = \theta_1 - \theta_0$ and, in the last steps, we have used the facts that $\Phi(-x) = 1 - \Phi(x)$ and $z_{1-\alpha} = -z_\alpha$.

- The preceding computations lead to

$$\sqrt{n}\frac{\Delta}{\sigma} + z_\alpha \geq z_{1-\beta}$$
$$\implies \sqrt{n}\frac{\Delta}{\sigma} \geq (z_{1-\beta} - z_\alpha) = -(z_\alpha + z_\beta)$$
$$\implies n \geq (z_\alpha + z_\beta)^2 \left(\frac{\sigma}{\Delta}\right)^2.$$

- Thus, we arrive at the ubiquitous sample size formula:

The sample size formula:

$$n = (z_\alpha + z_\beta)^2 \left(\frac{\sigma}{\Delta}\right)^2.$$

- For two-sided alternatives, one simply replaces α by $\alpha/2$ in the above expression.

- Suppose, we take the prior $\theta \sim N(\theta_1, \tau^2)$. We let $\tau^2 = \frac{\sigma^2}{n_0}$, where n_0 (*prior sample size!*) reflects the precision of the prior relative to the data. This simplifies the calculations:

$$N\left(\theta \,|\, \theta_1, \frac{\sigma^2}{n_0}\right) \times N\left(\bar{y} \,|\, \theta, \frac{\sigma^2}{n}\right)$$
$$= N\left(\theta \,|\, \frac{n_0}{n+n_0}\theta_1 + \frac{n}{n+n_0}\bar{y}, \frac{\sigma^2}{n+n_0}\right).$$

- Let $A_\alpha(\theta_0, \theta_1) = \{\bar{y} : P(\theta < \theta_0 \,|\, \bar{y}) < \alpha\}$. Note:

$$P(\theta < \theta_0 \,|\, \bar{y}) =$$
$$P\left\{\frac{\sqrt{n+n_0}}{\sigma}\left(\theta - \frac{n_0\theta_1 + n\bar{y}}{n+n_0}\right) < \frac{\sqrt{n+n_0}}{\sigma}\left(\theta_0 - \frac{n_0\theta_1 + n\bar{y}}{n+n_0}\right)\right\}$$
$$= \Phi\left[\frac{\sqrt{n+n_0}}{\sigma}\left(\theta_0 - \frac{n_0\theta_1 + n\bar{y}}{n+n_0}\right)\right]$$

- Thus,

$$
\begin{aligned}
A_\alpha(\theta_0, \theta_1) &= \left\{ \bar{y} : \frac{\sqrt{n + n_0}}{\sigma} \left(\theta_0 - \frac{n_0\theta_1 + n\bar{y}}{n + n_0} \right) < z_\alpha \right\} \\
&= \left\{ \bar{y} : \theta_0 - \frac{n_0\theta_1 + n\bar{y}}{n + n_0} < \frac{\sigma}{\sqrt{n + n_0}} z_\alpha \right\} \\
&= \left\{ \bar{y} : \theta_0 - \frac{n_0}{n + n_0}\theta_1 - \frac{n}{n + n_0}\bar{y} < \frac{\sigma}{\sqrt{n + n_0}} z_\alpha \right\} \\
&= \left\{ \bar{y} : \bar{y} > \theta_0 - \frac{n_0}{n}(\theta_1 - \theta_0) - \sqrt{\left(1 + \frac{n_0}{n}\right)} \frac{\sigma}{\sqrt{n}} z_\alpha \right\}
\end{aligned}
$$

- Note that as $n_0 \rightarrow 0$ (i.e. the prior becomes vague) $A_\alpha(\theta_0, \theta_1)$ becomes identical to the the critical region from classical hypothesis testing.

- The Bayesian power or Bayesian assurance δ is defined as:

Bayesian assurance:

$$\begin{aligned}
\delta &= P_{\bar{y}}(A_\alpha(\theta_0, \theta_1)) \\
&= P_{\bar{y}}\{\bar{y} : P(\theta < \theta_0 \,|\, \bar{y}) < \alpha\} \\
&= P_{\bar{y}}\{\bar{y} : P(\theta > \theta_0 \,|\, \bar{y}) > 1 - \alpha\} \\
&= P_{\bar{y}}\left\{\bar{y} > \theta_0 - \frac{n_0}{n}(\theta_1 - \theta_0) - \sqrt{\left(1 + \frac{n_0}{n}\right)}\frac{\sigma}{\sqrt{n}} z_\alpha\right\}
\end{aligned}$$

- The marginal distribution of $\bar{y}$ is given by:

$$\int N\left(\theta \,|\, \theta_1, \frac{\sigma^2}{n_0}\right) \times N\left(\bar{y} \,|\, \theta, \frac{\sigma^2}{n}\right) d\theta = N\left(\bar{y} \,|\, \theta_1, \frac{\sigma^2}{n + n_0}\right).$$

- Therefore, the Bayesian power or assurance is:

$$\begin{aligned}
&P_{\bar{y}}\left\{\bar{y} > \theta_0 - \frac{n_0}{n}(\theta_1 - \theta_0) - \frac{\sqrt{n+n_0}}{n}\sigma z_\alpha\right\} \\
&= P_{\bar{y}}\left\{\bar{y} - \theta_1 > -\left(1 + \frac{n_0}{n}\right)(\theta_1 - \theta_0) - \frac{\sqrt{n+n_0}}{n}\sigma z_\alpha\right\} \\
&= P\left(Z > -\sqrt{n+n_0}\left(1 + \frac{n_0}{n}\right)\frac{\theta_1 - \theta_0}{\sigma} - \left(1 + \frac{n_0}{n}\right) z_\alpha\right) \\
&= \Phi\left(\sqrt{n+n_0}\left(1 + \frac{n_0}{n}\right)\frac{\theta_1 - \theta_0}{\sigma} + \left(1 + \frac{n_0}{n}\right) z_\alpha\right)
\end{aligned}$$

- Rewriting the Bayesian power in terms of the relative precision n_0/n and n, we obtain:

$$\delta = \Phi\left(\sqrt{n}\left(1+\frac{n_0}{n}\right)^{3/2}\frac{\Delta}{\sigma} + \left(1+\frac{n_0}{n}\right)z_\alpha\right),$$

  where $\Delta = \theta_1 - \theta_0$. This is often called the *critical difference* that needs to be detected.

- This leads to:

The Bayesian Power (Assurance) Curve

$$\delta(\Delta, n) = \Phi\left(\sqrt{n}\left(1+\frac{n_0}{n}\right)^{3/2}\left(\frac{\Delta}{\sigma}\right) + \left(1+\frac{n_0}{n}\right)z_\alpha\right),$$

- Note: For study design purposes, σ^2 and n_0 are assumed known and the Bayesian power curve is investigated as a function of Δ and n.

- Given n_0, the Bayesian will compute the sample size needed to detect a critical difference of Δ with probability $1-\beta$ as
$$n = \arg\min\{n : \delta(\Delta, n) \geq 1-\beta\}$$
- As the prior becomes vague, $n_0 \to 0$ and:
$$\lim_{n_0 \to 0} \delta(\Delta, n) = \Phi\left(\sqrt{n}\frac{\Delta}{\sigma} + z_\alpha\right)$$
which is exactly the classical power curve. Now the Bayesian sample size formula coincides with the classical sample size formula:

Bayesian sample size with vague prior information:

$$n = (z_\alpha + z_\beta)^2 \left(\frac{\sigma}{\Delta}\right)^2.$$