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## Large Point-referenced Datasets

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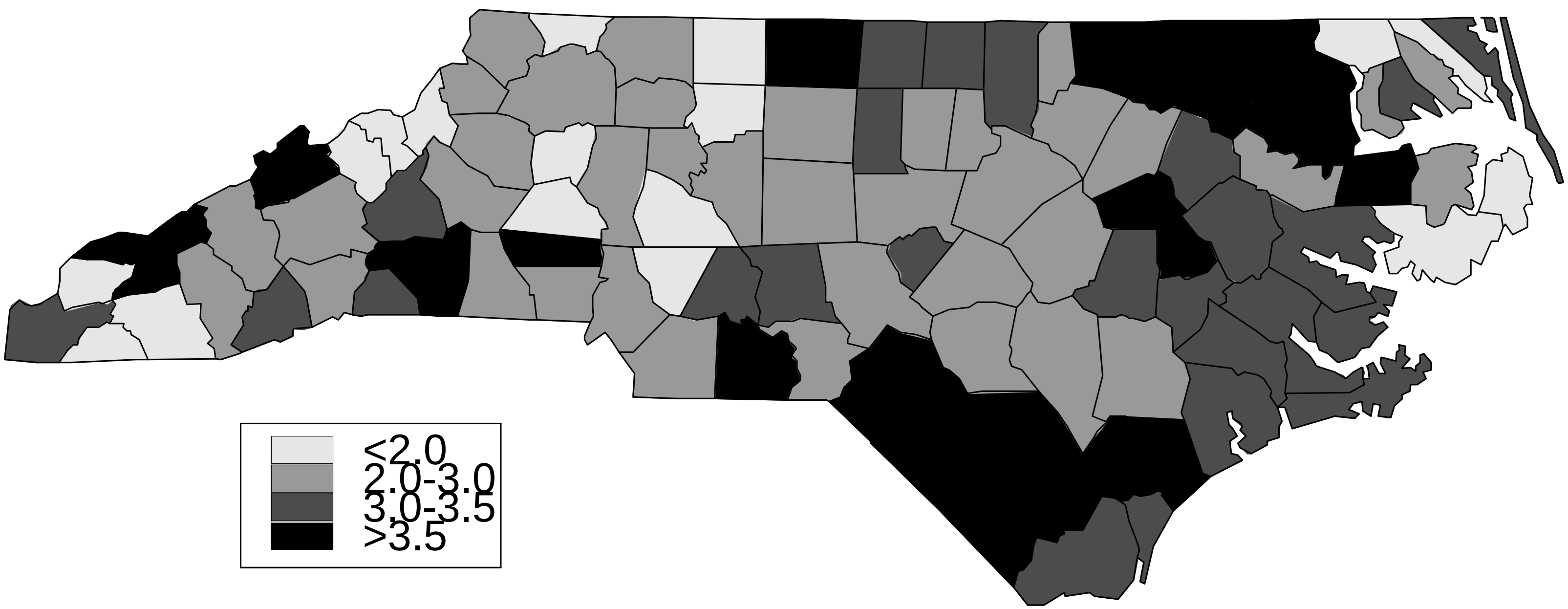
- Inference for new areal units?
- Descriptive/algorithmic vs. Model-based

# Areal unit data

## Actual Transformed SIDS Rates



# Proximity matrices

- $W$, entries $w_{ij}$ (with $w_{ii} = 0$). Choices for $w_{ij}$:
  - $w_{ij} = 1$ if $i, j$ share a common boundary (possibly a common vertex)
  - $w_{ij}$ is an *inverse* distance between units
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- $W$ elements often called "weights"; interpretation
- Could also define first-order neighbors $W^{(1)}$, second-order neighbors $W^{(2)}$, etc.

# Measures of spatial association

- Moran's $I$: essentially an "areal covariogram"

$$I = \frac{n \sum_i \sum_j w_{ij}(Y_i - \bar{Y})(Y_j - \bar{Y})}{(\sum_{i \neq j} w_{ij}) \sum_i (Y_i - \bar{Y})^2}$$

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- Both are asymptotically normal if $Y_i$ are i.i.d.; Moran has mean $-1/(n-1) \approx 0$, Geary has mean 1
- Significance testing by comparing to a collection of say 1000 random permutations of the $Y_i$

# Measures of spatial association (cont'd)

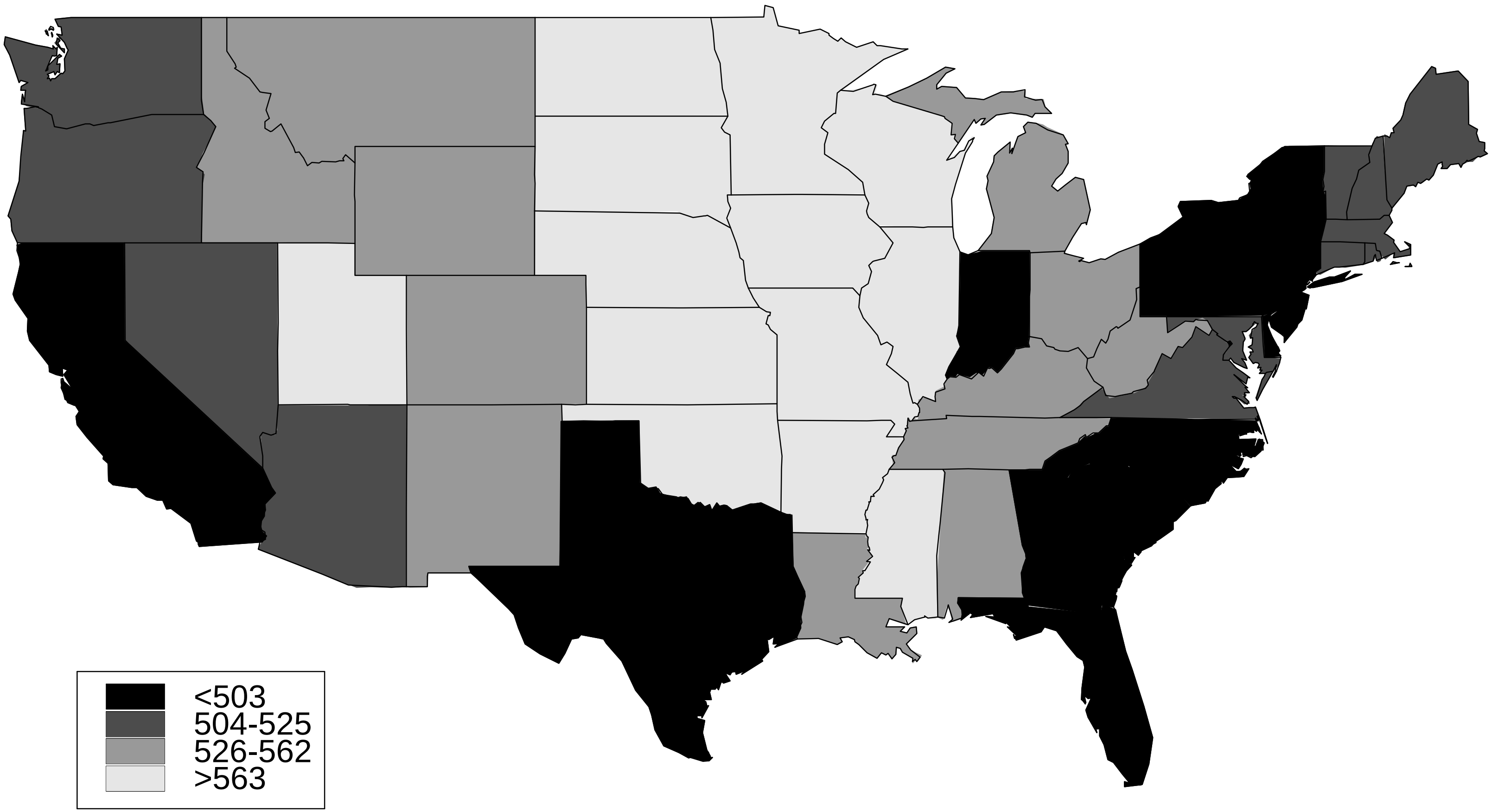


Figure 1: Choropleth map of 1999 average verbal SAT scores, lower 48 U.S. states.

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- For these data, the `spatial.cor` function in `S+SpatialStats` gives a Moran's $I$ of 0.5833, with associated standard error estimate 0.0920 $\Rightarrow$ very strong evidence against $H_0$ : no spatial correlation

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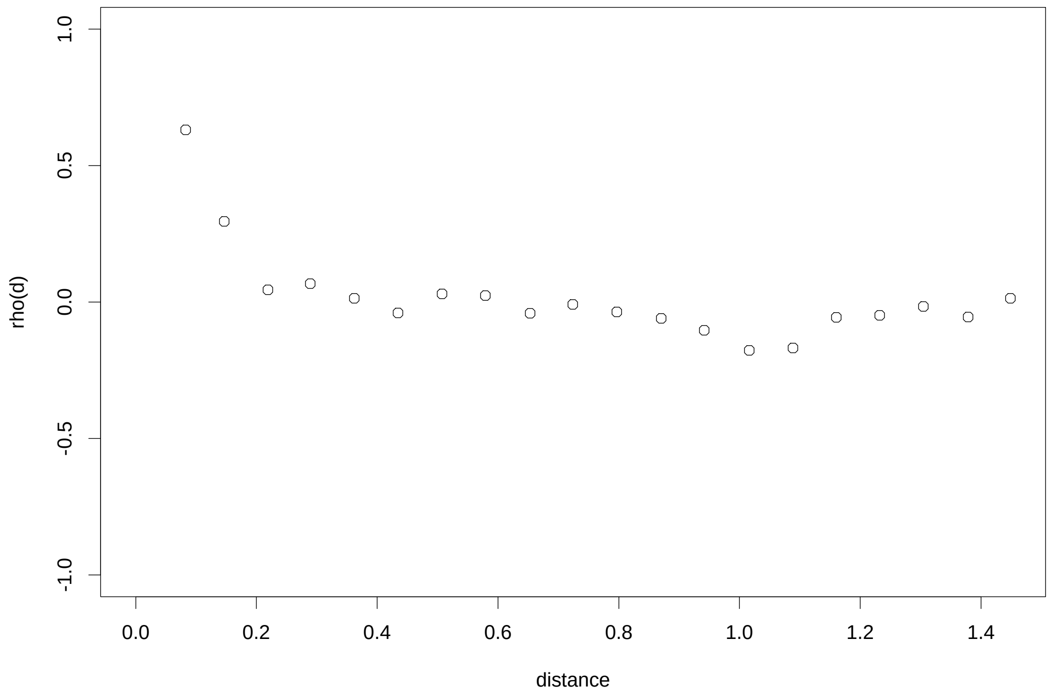
- $\Rightarrow$ the map, $I$, and $C$ all motivate the search for spatial covariates!

# Correlogram (via Moran's $I$)

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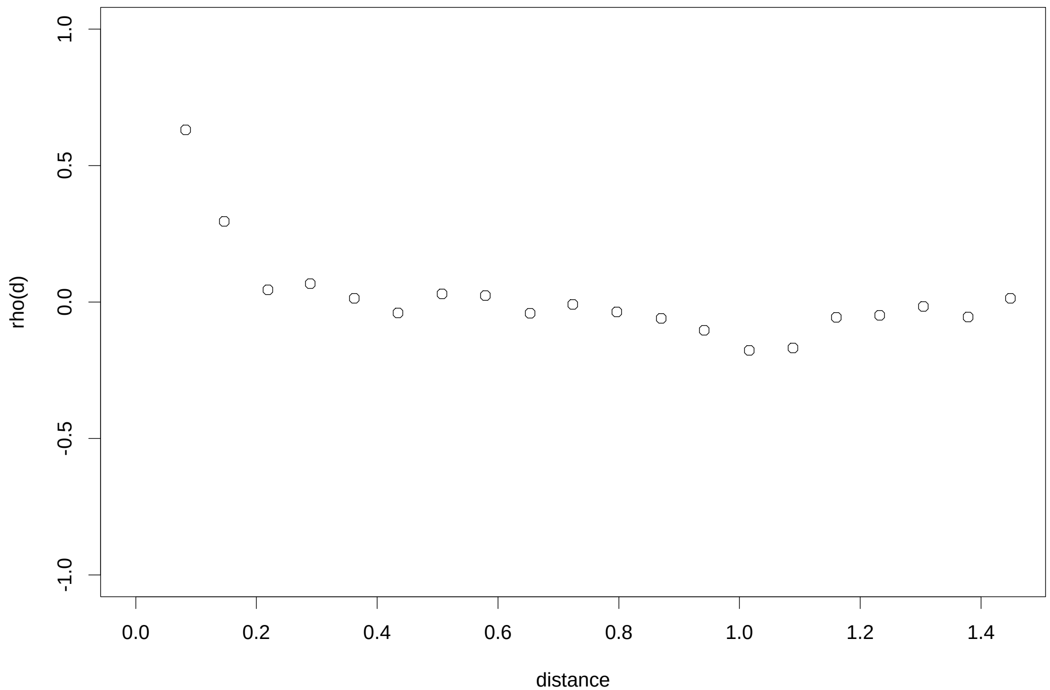


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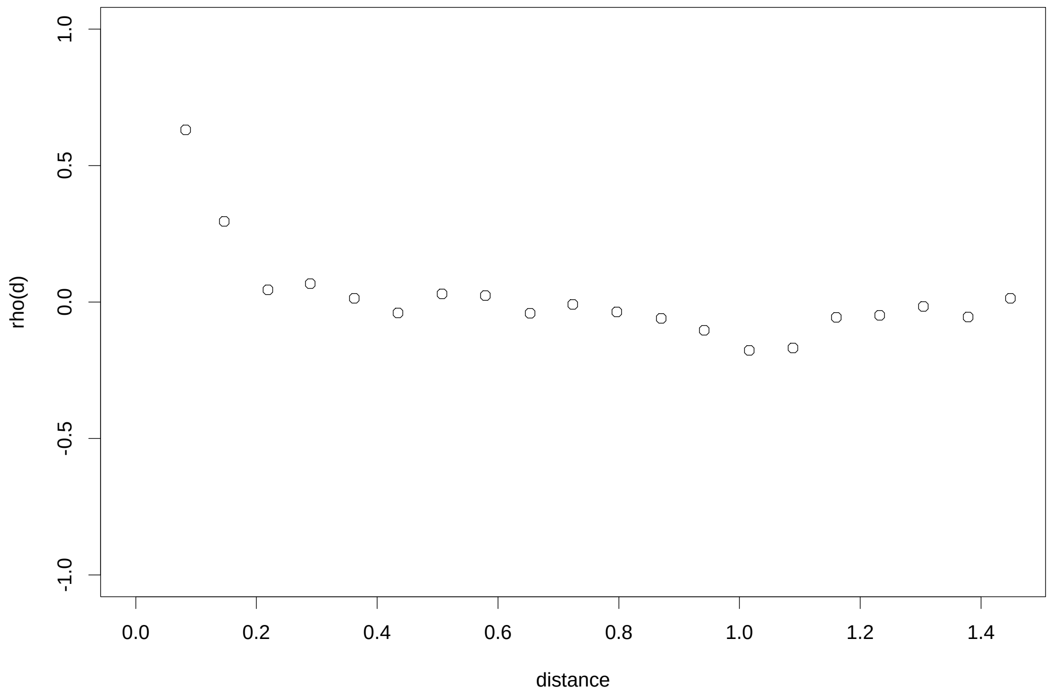


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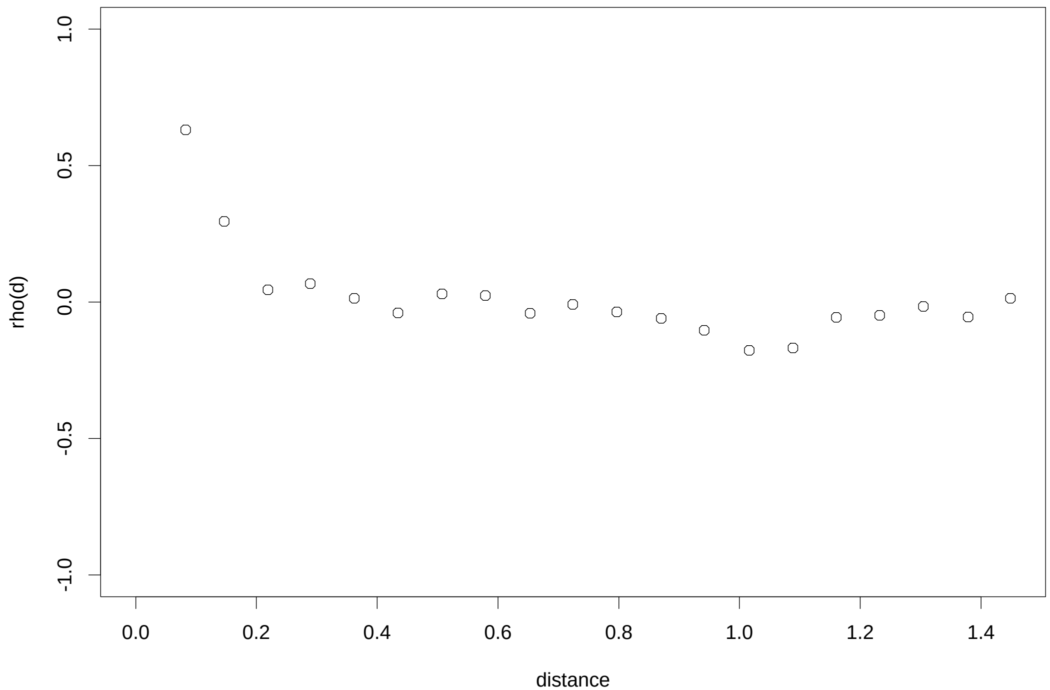


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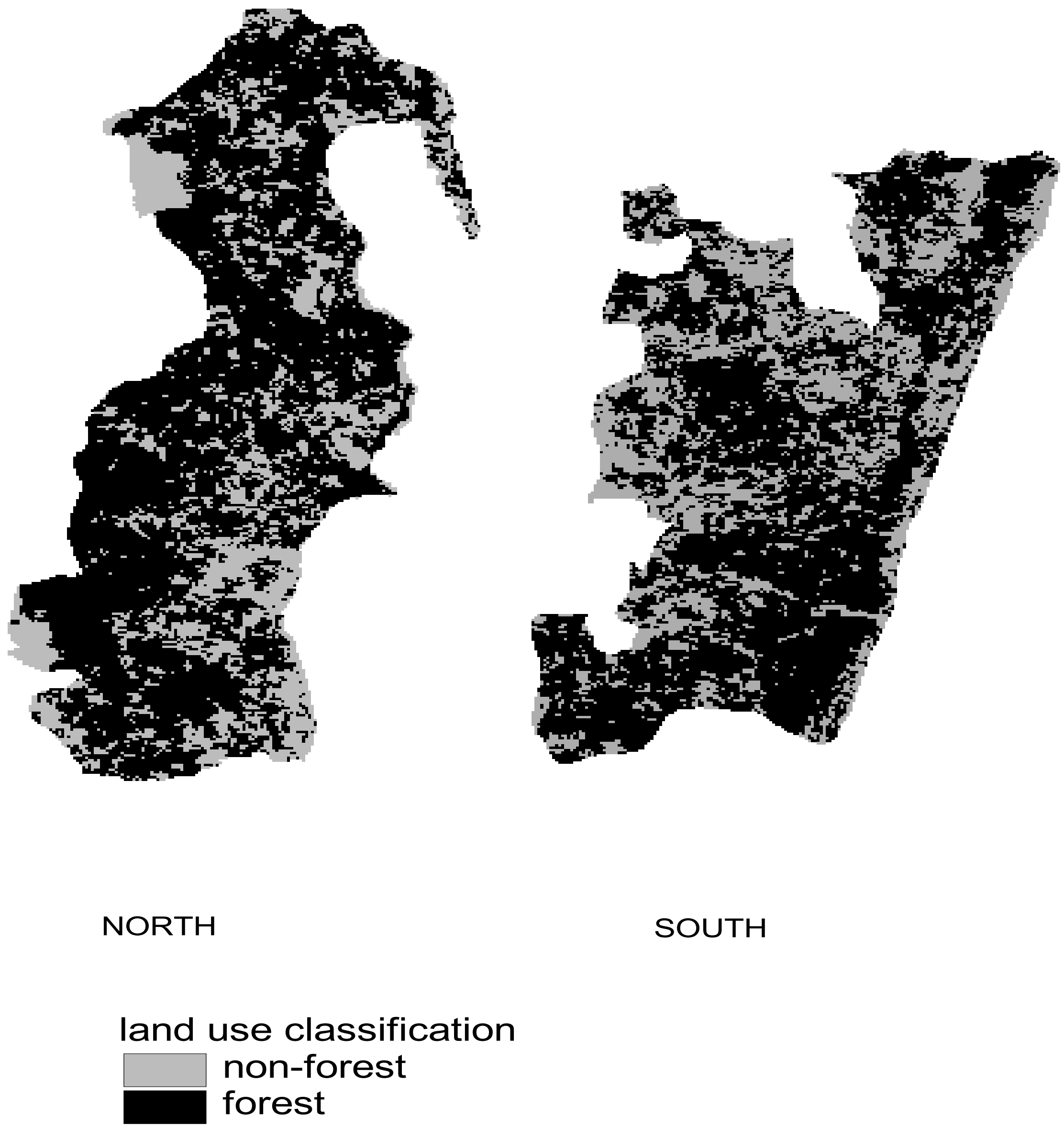
- spatial analogue of the temporal lag autocorrelation plot

# Rasterized binary data map



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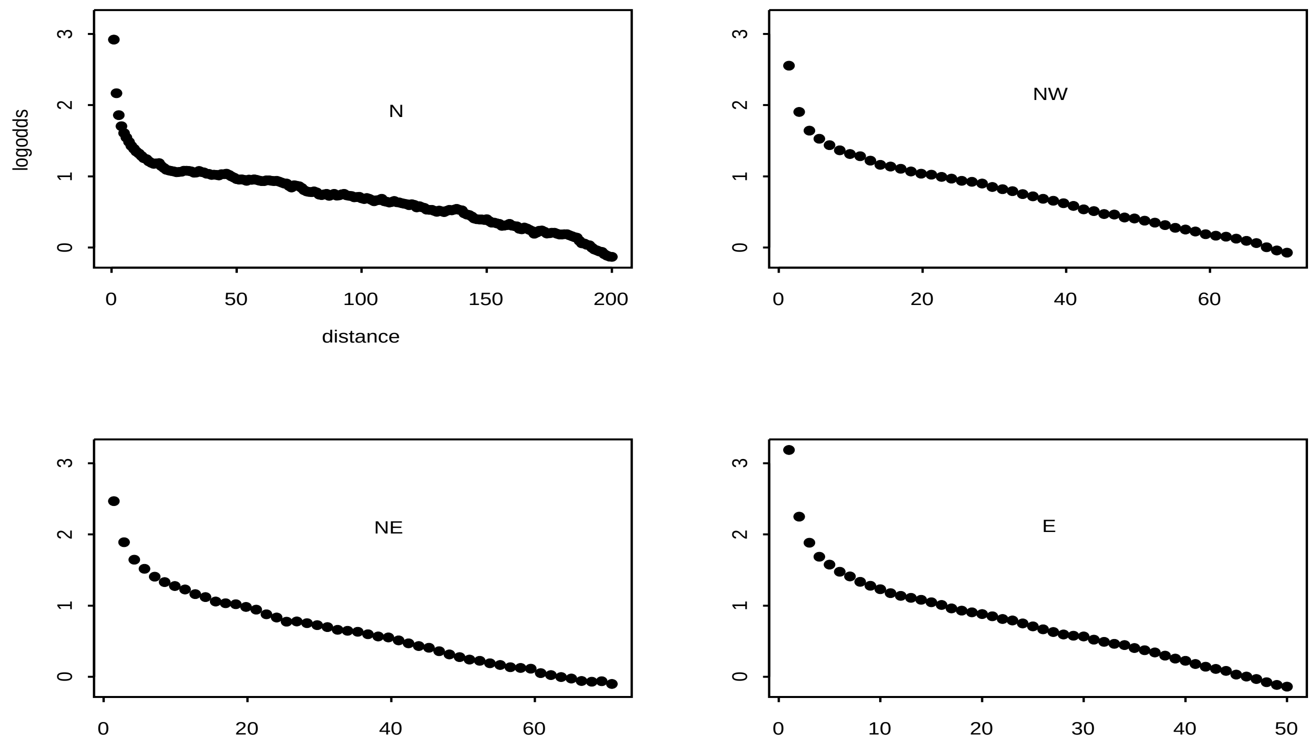


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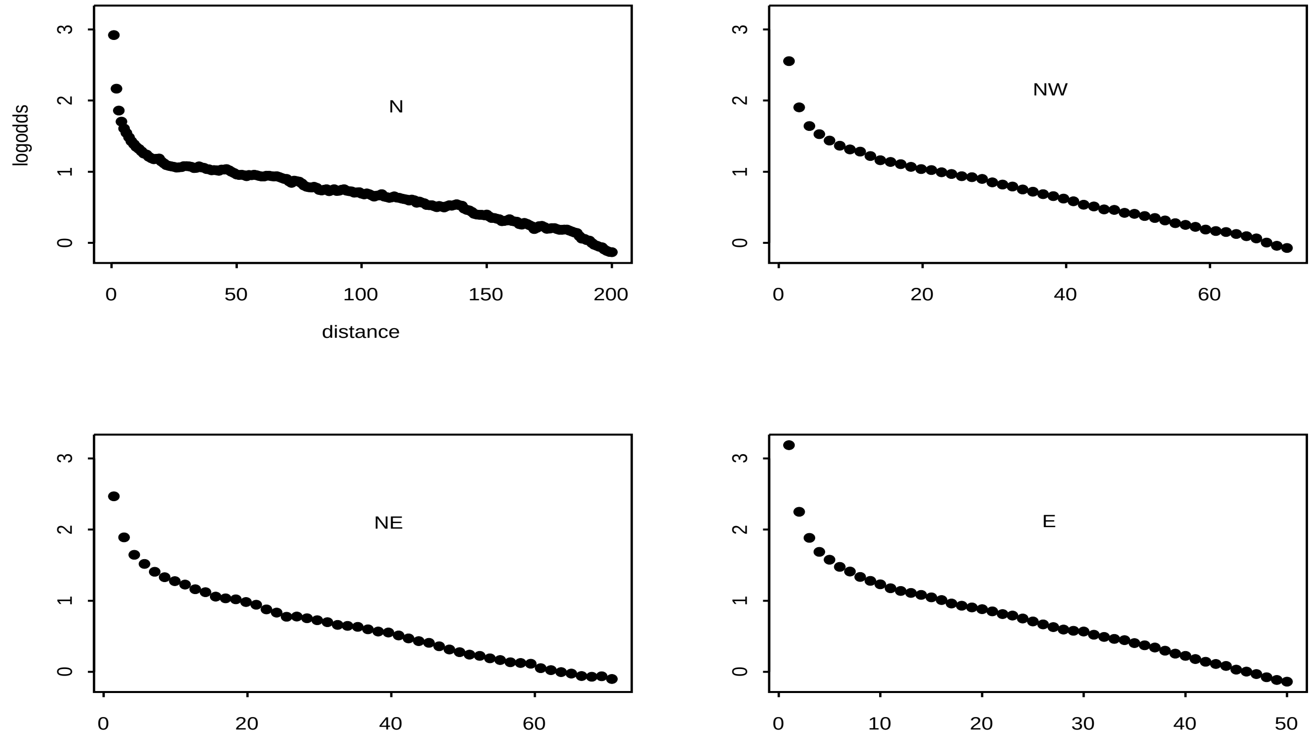


- A version for a binary map, using two-way tables and log odds ratios at the pixel level
- Note strongest pattern is to the north (N), but in no direction are the values $\approx$ 0 even at 40 km

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- Finally, we could try model-based smoothing, i.e., based on $E(Y_i|Data)$, i.e., the mean of the predictive distribution. Smoothers then emerge as byproducts of the hierarchical spatial models we use to explain the $Y_i$'s

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- Does $\{p(y_i|y_j, j \neq i)\}$ determine $p(y_1, y_2, ...y_n)$???
- If the full conditionals are compatible, then by Brook's Lemma,

$$p(y_1, \ldots, y_n) = \frac{p(y_1|y_2, \ldots, y_n)}{p(y_{10}|y_2, \ldots, y_n)} \frac{p(y_2|y_{10}, y_3, \ldots, y_n)}{p(y_{20}|y_{10}, y_3, \ldots, y_n)} \cdots \frac{p(y_n|y_{10}, \ldots, y_{n-1,0})}{p(y_{n0}|y_{10}, \ldots, y_{n-1,0})} p(y_{10}, \ldots, y_{n0})$$

If left side is proper, the fact that it integrates to 1 determines the normalizing constant!

# “Local” modeling

- Suppose we specify the full conditionals such that

$$p(y_i|y_j, j \neq i) = p(y_i|y_j \in \partial_i) \ ,$$

where $\partial_i$ is the set of neighbors of cell (region) $i$.
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- Example: For binary data, $k = 2$, we might take $Q(y_i, y_j) = I(y_i = y_j) = y_i y_j + (1 - y_i)(1 - y_j)$

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- Geman and Geman result : If we have a joint Gibbs distribution, then we have a Markov Random Field

# Conditional autoregressive (CAR) model

- Gaussian (autonormal) case

$$p(y_i|y_j, j \neq i) = N\left(\sum_j b_{ij} y_j, \tau_i^2\right)$$

Using Brook's Lemma we can obtain

$$p(y_1, y_2, ...y_n) \propto \exp\left\{-\frac{1}{2}\mathbf{y}' D^{-1}(I - B)\mathbf{y}\right\}$$

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- $\Rightarrow$ suggests a multivariate normal distribution with $\mu_Y = 0$ and $\Sigma_Y = (I - B)^{-1}D$
- $D^{-1}(I - B)$ symmetric requires $\frac{b_{ij}}{\tau_i^2} = \frac{b_{ji}}{\tau_j^2}$ for all $i, j$

# CAR Model (cont'd)

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- Not a data model – a random effects model!

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"To $\rho$ or not to $\rho$?" (that is the question)

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- So, used with random effects, scope of spatial pattern may be limited

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- Non-Gaussian case: For binary data, the autologistic:

$$p(y_i|y_j, j \neq i) \propto \exp\left\{\phi \sum_j w_{ij} I(y_i = y_j)\right\}$$