PubH 7405:
REGRESSION ANALYSIS

A REVIEW OF MATRICES

Scientists use statistics in their research – for the design and analysis of their studies. They consult statisticians when needed; and, in doing so, they need to understand some statistical reasoning that underlies the method they employ.

Statisticians are concerned with the applications of existing statistical methods – and also with devising new methods from time to time; and, in doing so, they need to know enough mathematics for the tasks. One of such areas of mathematics is "matrices".

The roles of mathematics in statistics is similar to that of statistics in science as a whole.

Statistics is the “language” for scientists and Mathematics is the “language” for statisticians.

A "**matrix**" is a display of numbers or numerical quantities laid out in a **rectangular array of rows and columns**. The array, or **two-way table of numbers**, could be **rectangular or square** – could be just **one row** (a row matrix or row vector) or **one column** (a column matrix or column vector). When it is square, it could be **symmetric** or a "**diagonal matrix**" (non-zero entries are on the main diagonal). The numbers of rows and of columns form the "dimension" of a matrix; for example, the dimension of the following matrix is "3x2" – **three rows and two columns**.

$$\begin{bmatrix} 16{,}000 & 23 \\ 33{,}000 & 47 \\ 21{,}000 & 35 \end{bmatrix}$$

A number is the most simple matrix; a 1x1 matrix, one row one column – called a "scalar"

An **“entry” or “element” of a matrix** need two subscripts for identification; **the first for the row number and the second for the column number:**

$$\mathbf{A} = [a_{ij}]$$

For example, in the following matrix we have $a_{11} = 16{,}000$ and $\mathbf{a_{32} = 35}$.

$$\mathbf{A} = \begin{bmatrix} 16{,}000 & 23 \\ 33{,}000 & 47 \\ 21{,}000 & 35 \end{bmatrix}$$

Two matrices A and B are equal if and only if they have the **same dimension** and **all corresponding elements are equal** – **position by position**

$$\mathbf{A} = [a_{ij}]_{rxc}$$

$$\mathbf{B} = [b_{ij}]_{nxm}$$

$$\mathbf{A} = \mathbf{B} \Leftrightarrow \{r = n, c = m, a_{ij} = b_{ij} \text{ for all i and all j}\}$$

Again, a "**vector**" is a matrix with only one row (a **row vector**) or with only one column (a **column vector**). It is conventional to assume that all vectors are column vectors – unless otherwise stated.

$$\mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

A “**triangular matrix**” is a square matrix with all the elements above, or below, the main diagonal equal to 0; called “lower triangular matrix” and “upper triangular matrix”. For example,

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 7 & -6 & 0 \\ 9 & 22 & 11 \end{bmatrix}$$

A **square** matrix whose elements are all zero except for those on the main diagonal is called a diagonal matrix. If all diagonal elements of a diagonal matrix are **equal**, the matrix is call a "**scalar matrix**":

$$\mathbf{D} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

In particular, a scalar matrix with the common diagonal element equal to 1 is a “**unit matrix**” and is denoted by “**I**” with a subscript specifying its dimension (number of rows/columns). For example:

$$\mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The notation **J** is used for a square matrix with all elements equal 1

A 1x1 matrix is "square", "symmetric", "triangular", "diagonal", and "scalar"; but it may or may not be a "unit matrix".

Any matrix all of whose elements are equal to zero is a "zero matrix"; for example, 0_{3x4}. A zero matrix is scalar matrix.

$$\mathbf{0}_{3x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

TRANSPOSE

The transpose of a matrix **A** is another matrix, denoted by **A'** (or $\mathbf{A}^T$), that is obtained by interchanging the columns and the rows of the matrix **A**; that is:

$$\text{If } \mathbf{A} = \left[a_{ij}\right] \text{ then } \mathbf{A}' = \left[a_{ji}\right]$$

$$\mathbf{A}_{3x2} = \begin{bmatrix} 2 & 5 \\ 7 & 10 \\ 3 & 4 \end{bmatrix}; \mathbf{A}'_{2x3} = \begin{bmatrix} 2 & 7 & 3 \\ 5 & 10 & 4 \end{bmatrix}$$

If **A** is a symmetric matrix, $a_{ij} = a_{ji}$, we have $\mathbf{A} = \mathbf{A}'$

The roles of rows and columns are swapped

The transpose of a column vector is a row vector

$$\mathbf{C} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 12 \end{bmatrix}; \mathbf{C}' = \begin{bmatrix} 3 & -2 & 0 & 12 \end{bmatrix}$$

REGRESSION EXAMPLES

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ & \vdots \\ 1 & x_n \end{bmatrix}$$

X is referred to as the **"Design Matrix"**

In the "**regression through the origin**", the **Design Matrix X has only one column** – the column with predictor's values.

ADD & SUBTRACT MATRICES

Adding or subtracting 2 matrices requires that they have the **same dimensions**. The sum (or difference) of 2 matrices is another matrix whose elements each consist of the sum (or difference) of the corresponding elements of the two matrices – position by position.

$$\mathbf{A}_{rxc} = [a_{ij}] \text{ and } \mathbf{B}_{rxc} = [b_{ij}]$$

$$\text{Then } \mathbf{A} \pm \mathbf{B} = [a_{ij} \pm b_{ij}]$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1+1=2 & 4+2=6 \\ 2+2=4 & 5+3=8 \\ 3+3=6 & 6+4=10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 \\ 4 & 8 \\ 6 & 10 \end{bmatrix}$$

EXAMPLE: Regression Model

$$\mathbf{Y} = \mathbf{E}(\mathbf{Y}) + \boldsymbol{\varepsilon}$$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} E(Y_1) \\ E(Y_2) \\ \vdots \\ E(Y_n) \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_{\mathrm{n}} \end{bmatrix}$$

$$= \begin{bmatrix} E(Y_1) + \varepsilon_1 \\ E(Y_2) + \varepsilon_2 \\ \vdots \\ E(Y_n) + \varepsilon_{\mathrm{n}} \end{bmatrix}$$

MULTIPLICATION by a Scalar

- Again, “scalar” is an (ordinary) number
- In multiplying a matrix by a scalar, every element of the matrix is multiplied by that scalar
- The result is a new matrix with the **same dimension**

$$\mathbf{A} = [a_{ij}]$$

$$k\mathbf{A} = [ka_{ij}]$$

MULTIPLICATION of Matrices

- Multiplication of a matrix by another matrix is much more complicated.
- First, the "order" is important; "**A**x**B**" is said to be "**A** is post-multiplied by **B**" or "**B** is pre-multiplied by **A**"; In general: **A**x**B** ≠ **B**x**A**
- There is a strong requirement on the dimensions: **A**x**B** is defined only if "the number of columns of **A** is equal to the number of rows of **B**".
- The product **A**x**B** has, as its dimension, the number of rows of **A** and the number of columns of **B**.

$$\mathbf{A}_{rxc}\mathbf{B}_{cxs} = \mathbf{AB}_{rxs}$$

For example:

$$\mathbf{A}_{2\mathrm{x}3}\mathbf{B}_{3\mathrm{x}2} = \mathbf{AB}_{2\mathrm{x}2}$$

The common dimension, c, disappears

MULTIPLICATION FORMULA

$$\mathbf{A}_{rxc}\mathbf{B}_{cxs} = \mathbf{AB}_{rxs}$$

$$[a_{ij}]_{rxc}[b_{ij}]_{cxs} = [\sum_{k=1}^{c} a_{ik}b_{kj}]_{rxs}$$

For entry (i,j) of **A**x**B**, we multiply row i of **A** by column j of **B**.

EXAMPLES

$$\begin{bmatrix} 4 & 2 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4a_1 + 2a_2 \\ 5a_1 + 8a_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2^2 + 3^2 + 5^2 = 38 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 33 & 52 \\ 21 & 32 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 33 & 52 \\ 21 & 32 \end{bmatrix}$$

Note: For entry/cell (1,2) of the product,

$$\begin{bmatrix} 2 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = (2)(6) + (5)(8) = 12 + 40 = 52$$

It is interesting to note the effects of multiplying on the left or right by a scalar matrix:

$$\begin{bmatrix} 3 & 12 & 9 \\ 0 & -7 & 5 \\ 14 & 2 & 21 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 12 & 48 & 36 \\ 0 & -28 & 20 \\ 56 & 8 & 84 \end{bmatrix}$$

Note: Compare to: $\mathbf{A} = [a_{ij}]$

$$k\mathbf{A} = [ka_{ij}]$$

REGRESSION EXAMPLES

$$\mathbf{Y}'\mathbf{Y} = [y_1 \quad y_2 \quad \cdots \quad y_n] \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix} = [\sum y_i^2]$$

$$\mathbf{X}'\mathbf{Y} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

MORE REGRESSION EXAMPLE

$$\mathbf{X'X} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ & \vdots \\ 1 & x_n \end{bmatrix}$$

$$= \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}$$

Again, **X** is referred to as the “Design Matrix”

THREE IMPORTANT MATRICES IN REGRESSION ANALYSIS

$$\mathbf{Y}'\mathbf{Y} = \left[\sum y_i^2\right]$$

$$\mathbf{X}'\mathbf{Y} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}$$

SIMPLE LINEAR REGRESSION MODEL

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i ; i = 1,2,\ldots,n$$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$\mathbf{Y}_{nx1} = \mathbf{X}_{nx2} \boldsymbol{\beta}_{2x1} + \boldsymbol{\varepsilon}_{nx1}$$

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

$$= \begin{bmatrix} \beta_0 + \beta_1 X_1 + \varepsilon_1 \\ \beta_0 + \beta_1 X_2 + \varepsilon_2 \\ \vdots \\ \beta_0 + \beta_1 X_n + \varepsilon_n \end{bmatrix}$$

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

BASIC RESULTS #1

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
- $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
- $\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$
- $k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$
- $(\mathbf{A}')' = \mathbf{A}$
- $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$
- $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$
- $(\mathbf{ABC})' = \mathbf{C}'\mathbf{B}'\mathbf{A}'$

Pay attention to the last two equations

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 4 & 6 \\ 5 & 8 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 33 & 52 \\ 21 & 32 \end{bmatrix}$$

$$(\mathbf{AB})' = \begin{bmatrix} 33 & 21 \\ 52 & 32 \end{bmatrix}$$

$$\mathbf{B}'\mathbf{A}' = \begin{bmatrix} 4 & 5 \\ 6 & 8 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 33 & 21 \\ 52 & 32 \end{bmatrix}$$

LINEAR DEPENDENCE

Consider a set of c column vectors $\mathbf{C_1}$, $\mathbf{C_2}$, …, $\mathbf{C_c}$ in a rxc matrix. If we can find c scalars k_1, k_2, …, k_c – not all zero – so that:

$$k_1\mathbf{C_1} + k_2\mathbf{C_2} + \ldots + k_c\mathbf{C_c} = \mathbf{0}$$

the c column vectors are said to be "linearly dependent". If the only set of scalars for which the above equation holds is all zero ($k_1 = \ldots = k_c = 0$), the c column vectors is **linearly independent**.

EXAMPLE

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 7 & 21 \end{bmatrix}$$

Since :

$$(3)\begin{bmatrix} 2 \\ 7 \end{bmatrix} + (-1)\begin{bmatrix} 6 \\ 21 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the two column vectors are linearly dependent

RANK OF A MATRIX

- The **rank** of a matrix is defined as "the **maximum number of linearly independent columns** in the matrix.
- In the previous example, the rank of **A**<4
- Since the first, second and fourth columns are linearly independent (no scalars can be found so that $k_1\mathbf{C_1} + k_2\mathbf{C_2} + k_4\mathbf{C_4} = \mathbf{0}$); the rank of that matrix **A** is 3.

SINGULAR/NON-SINGULAR MATRICES

- If the rank of a square r x r matrix **A** is r then matrix A is said to be nonsingular or of "full rank"
- An r x r matrix with rank less than r is said to be singular or not of full rank.

INVERSE OF A MATRIX

- The inverse of a matrix **A** is another matrix $\mathbf{A}^{-1}$ such that: $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ where **I** is the identity or unit matrix.
- An inverse of a matrix is defined **only for square matrices.**
- Many matrices do not have an inverse; a singular matrix does not have an inverse.
- If a square matrix has an inverse, the inverse is unique; the inverse of a nonsingular or full rank matrix is also nonsingular and has same rank.

EXAMPLE

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -.1 & .4 \\ .3 & -.2 \end{bmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} -.1 & .4 \\ .3 & -.2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -.1 & .4 \\ .3 & -.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A SPECIAL EXAMPLE

$$\mathbf{A}=\begin{bmatrix}3&0&0\\0&4&0\\0&0&2\end{bmatrix};\ \mathbf{A}^{-1}=\begin{bmatrix}1/3&0&0\\0&1/4&0\\0&0&1/2\end{bmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A}=\begin{bmatrix}1/3&0&0\\0&1/4&0\\0&0&1/2\end{bmatrix}\begin{bmatrix}3&0&0\\0&4&0\\0&0&2\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Note: Look at the diagonals on first row

BASIC RESULTS #2

- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$

Pay attention to the last two equations

FINDING THE INVERSE

Finding the inverse of a matrix often requires a large amount of computing – except for diagonal matrices; we give here only formulas for 2x2 and 3x3matrices – others can be seen as defined similarly.

2x2 MATRICES

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{d}{D} & \frac{-b}{D} \\ \frac{-c}{D} & \frac{a}{D} \end{bmatrix}$$

$$\mathbf{D = (ad - bc)\ is\ the\ "determinant"\ of\ A;\ or\ |\ A\ |}$$

EXAMPLE

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}; \mathrm{a} = 2, \mathrm{b} = 4, \mathrm{c} = 3, \text{and } \mathrm{d} = 1$$

$$D = ad - bc = (2)(1) - (3)(4) = -10$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{-10} & \frac{-4}{-10} \\ \frac{-3}{-10} & \frac{2}{-10} \end{bmatrix}$$

$$= \begin{bmatrix} -.1 & .4 \\ .3 & -.2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -.1 & .4 \\ .3 & -.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A **singular matrix does not have an inverse** because **its determinant is zero** and we cannot divide any number by zero.

$$\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 7 & 21 \end{bmatrix}$$

Since:

$$(3)\begin{bmatrix} 2 \\ 7 \end{bmatrix} + (-1)\begin{bmatrix} 6 \\ 21 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the two column vectors are linearly dependent,

matrix A is singular:

$$\mathrm{D} = (2)(21) - (7)(6) = 0$$

A SYSTEM OF EQUATIONS

It is extremely simple to write a system of equations in the matrix form – especially with many equations

$$3x + 4y - 10z = 0$$

$$-2x - 5y + 21z = 14$$

$$x + 29y - 2z = -3$$

$$\begin{bmatrix} 3 & 4 & -10 \\ -2 & -5 & 21 \\ 1 & 29 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 14 \\ -3 \end{bmatrix}$$

A SIMPLE APPLICATION

Consider a system of two equations:

$$2y_1 + 4y_2 = 20$$
$$3y_1 + y_2 = 10$$

which is written in matrix notation:

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} -.1 & .4 \\ .3 & -.2 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Solution:

$y_1 = 2$ & $y_2 = 4$

MORE EXAMPLE

$$\begin{bmatrix} 3 & 4 & -10 \\ -2 & -5 & 21 \\ 1 & 29 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 14 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 4 & -10 \\ -2 & -5 & 21 \\ 1 & 29 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 14 \\ -3 \end{bmatrix}$$

REGRESSION EXAMPLE

$$\mathbf{X'X} = \begin{bmatrix} n & \sum x \\ \sum x & \sum x^2 \end{bmatrix}$$

$$D = n\sum x^2 - (\sum x)(\sum x)$$

$$= n\sum (x - \bar{x})^2$$

REGRESSION EXAMPLE

$$\mathbf{X'X} = \begin{bmatrix} n & \sum x \\ \sum x & \sum x^2 \end{bmatrix}$$

$$D = n\sum (x - \bar{x})^2$$

$$(\mathbf{X'X})^{-1} = \begin{bmatrix} \dfrac{\sum x^2}{n\sum (x-\bar{x})^2} & \dfrac{-\sum x}{n\sum (x-\bar{x})^2} \\ \dfrac{-\sum x}{n\sum (x-\bar{x})^2} & \dfrac{n}{n\sum (x-\bar{x})^2} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{n} + \dfrac{\bar{x}^2}{\sum (x-\bar{x})^2} & \dfrac{-\bar{x}}{\sum (x-\bar{x})^2} \\ \dfrac{-\bar{x}}{\sum (x-\bar{x})^2} & \dfrac{1}{\sum (x-\bar{x})^2} \end{bmatrix}$$

FOUR IMPORTANT MATRICES IN REGRESSION ANALYSIS

$$\mathbf{Y}'\mathbf{Y} = [\sum y_i^2]$$

$$\mathbf{X}'\mathbf{Y} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$$\mathbf{X'X} = \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}$$

$$(\mathbf{X'X})^{-1} = \begin{bmatrix} \frac{1}{n} + \frac{\bar{x}^2}{\sum (x - \bar{x})^2} & \frac{-\bar{x}}{\sum (x - \bar{x})^2} \\ \frac{-\bar{x}}{\sum (x - \bar{x})^2} & \frac{1}{\sum (x - \bar{x})^2} \end{bmatrix}$$

LEAST SQUARE METHOD

$$\text{Data}: \ \{(x_i, y_i)\}_{i=1}^{n}$$

$$Q = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

$$\frac{\delta Q}{\delta \beta_0} = -2\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\frac{\delta Q}{\delta \beta_1} = -2\sum_{i=1}^{n} x_i (y_i - \beta_0 - \beta_1 x_i) = 0$$

NORMAL EQUATIONS

$$\frac{\delta Q}{\delta \beta_0} = \frac{\delta Q}{\delta \beta_1} = 0$$

Normal Equations:

$$\sum \mathrm{y_i} = nb_0 + b_1 \sum x_i$$

$$\sum \mathrm{x_i y_i} = b_0 \sum x_i + b_1 \sum x_i^2$$

$\text{In matrix notations:}$

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$$\mathbf{X'X}_{2x2}\mathbf{b}_{2x1} = \mathbf{X'Y}_{2x1}$$

LEAST SQUARE ESTIMATES

Normal Equations:

$$\sum \mathrm{y_i} = nb_0 + b_1 \sum x_i$$

$$\sum \mathrm{x_i y_i} = b_0 \sum x_i + b_1 \sum x_i^2$$

In matrix notations:

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$$\mathbf{(X'X)b = X'Y}$$

$$\mathbf{b} = (\mathbf{X'X})^{-1}(\mathbf{X'Y})$$

REGRESSION RESULTS

Normal Equations:

$$(\mathbf{X'X})\mathbf{b} = \mathbf{X'Y}$$

Least Square Estimates:

$$\mathbf{b} = (\mathbf{X'X})^{-1}(\mathbf{X'Y})$$

WE PROVE THE SAME RESULTS

$$\mathbf{X}'\mathbf{Y} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} \dfrac{1}{n} + \dfrac{\bar{x}^2}{\sum (x-\bar{x})^2} & \dfrac{-\bar{x}}{\sum (x-\bar{x})^2} \\ \dfrac{-\bar{x}}{\sum (x-\bar{x})^2} & \dfrac{1}{\sum (x-\bar{x})^2} \end{bmatrix}$$

$$\mathbf{B} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$

$$= \begin{bmatrix} \dfrac{1}{n} + \dfrac{\bar{x}^2}{\sum (x-\bar{x})^2} & \dfrac{-\bar{x}}{\sum (x-\bar{x})^2} \\ \dfrac{-\bar{x}}{\sum (x-\bar{x})^2} & \dfrac{1}{\sum (x-\bar{x})^2} \end{bmatrix} \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$$b_1 = \frac{\sum x_i y_i - \left[\sum x_i\right]\left[\sum y_i\right]/n}{\sum (x-\bar{x})^2}$$

Example #2:

SBP versus AGE

Age (x)	SBP (y)	x-sq	y-sq	xy
42	130	1764	16900	5460
46	115	2116	13225	5290
42	148	1764	21904	6216
71	100	5041	10000	7100
80	156	6400	24336	12480
74	162	5476	26244	11988
70	151	4900	22801	10570
80	156	6400	24336	12480
85	162	7225	26244	13770
72	158	5184	24964	11376
64	155	4096	24025	9920
81	160	6561	25600	12960
41	125	1681	15625	5125
61	150	3721	22500	9150
75	165	5625	27225	12375
Totals: 984	2193	67954	325929	146260

$$\mathbf{X'X} = \begin{bmatrix} n & \sum x \\ \sum x & \sum x^2 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 984 \\ 984 & 67{,}954 \end{bmatrix}$$

$$D = (15)(67{,}954) - (984)^2$$

$$= 51{,}054$$

$$(\mathbf{X'X})^{-1} = \begin{bmatrix} \frac{67{,}954}{51{,}064} & \frac{-984}{51{,}064} \\ \frac{-984}{51{,}064} & \frac{15}{51{,}064} \end{bmatrix}$$

$$= \begin{Bmatrix} 1.3310 & -.0193 \\ -.0193 & .0003 \end{Bmatrix}$$

<u>**Note:**</u> Rounding off errors; computer does better job

$$\mathbf{X'X} = \begin{bmatrix} 15 & 984 \\ 984 & 67{,}954 \end{bmatrix}$$

$$(\mathbf{X'X})^{-1} = \begin{bmatrix} 1.3310 & -.0193 \\ -.0193 & .0003 \end{bmatrix}$$

$$\mathbf{X'Y} = \begin{bmatrix} 2{,}193 \\ 146{,}260 \end{bmatrix}$$

$$\mathbf{b} = (\mathbf{X'X})^{-1}(\mathbf{X'Y})$$

$$= \begin{bmatrix} 1.3310 & -.0193 \\ -.0193 & .0003 \end{bmatrix} \begin{bmatrix} 2{,}193 \\ 146{,}260 \end{bmatrix}$$

$$= \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

Note: Results are sensitive to rounding off errors

Example #1: BIRTH WEIGHT DATA

	x (oz)	y (%)	x-sq	y-sq	xy
	112	63	12544	3969	7056
	111	66	12321	4356	7326
	107	72	11449	5184	7704
	119	52	14161	2704	6188
	92	75	8464	5625	6900
	80	118	6400	13924	9440
	81	120	6561	14400	9720
	84	114	7056	12996	9576
	118	42	13924	1764	4956
	106	72	11236	5184	7632
	103	90	10609	8100	9270
	94	91	8836	8281	8554
Totals	1207	975	123561	86487	94322

Can proceed similarly to calculate regression coefficients

Readings & Exercises

- Readings: A thorough reading of the text's sections 5.1-5.7 (pp.176-193) is highly recommended.
- Exercises: The following exercises are good for practice, all from chapter 5 of text: 5.1,5.2, and 5.7-5.11.
- **Due as Homework #12: 5.2 and 5.8.**