PubH 7401: Elements of Biostatistical Inference I – 2008
Homework 5, due Thursday, October 9

1. At a particular restaurant, the price of an entree is $E \sim N(10, 4)$ in U.S. dollars. That is, E is normally distributed with $\mu = 10$ and $\sigma^2 = 4$. Dessert is $D \sim N(4, 1)$ and a glass of wine is $W \sim N(6, 2)$.

   (a) What is the distribution of the total cost of dinner T for one at this restaurant, assuming independence among E, D, and W? Also assume that only one entree, one dessert, and one glass of wine will be consumed. **Answer**: $T = E + D + W$ and $T \sim N(10 + 4 + 6, 4 + 1 + 2)$, i.e. $T \sim N(20, 7)$.

   (b) If I only have \$10, what is the probability that I can afford dinner? That is, find $P(T \leq 10)$. **Answer**: $P(T \leq 10) = P(Z \leq -3.78) = 0.000082$, essentially zero. I guess I'll have to wash dishes.

2. Let (X, Y) be jointly distributed with pdf

$$f(x,y) = \begin{cases} 4xy & \text{for } \begin{array}{c} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{array} \\ 0 & \text{otherwise} \end{cases}$$

   (a) Does the joint pdf factor into $f(x, y) = g(x)h(y)$? **Answer**: $f(x, y) = [2x^2][2y^2] = f_X(x)f_Y(y)$. The joint factors into the two marginal distributions because the range of (X, Y) also factors $(0, 1) \times (0, 1)$.

   (b) Are X and Y independent? **Answer**: yes, because the joint $f(x, y)$ factors into the marginal densities and the range factors too.

   (c) Find the conditional pdf's $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$. **Answer**: Since X and Y are independent, $f_{X|Y}(x|y) = f_X(x) = 2x^2$ on $0 \leq x \leq 1$ and $f_{Y|X}(y|x) = f_Y(y) = 2y^2$ on $0 \leq y \leq 1$.

3. Let (X, Y) be jointly distributed with pdf

$$f(x,y) = \begin{cases} x+y & \text{for } \begin{array}{c} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{array} \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the marginal pdf's $f_X(x)$ and $f_Y(y)$. **Answer**:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 (x+y)dy = 0.5 + x \text{ on } 0 \leq x \leq 1.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^1 (x+y)dx = 0.5 + y \text{ on } 0 \leq y \leq 1.$$

   (b) Find the conditional pdf's $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$. **Answer**:

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{x+y}{0.5+y} \text{ on } 0 \leq x \leq 1.$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{x+y}{0.5+x} \text{ on } 0 \leq y \leq 1.$$

(c) Are X and Y independent? **Answer**: No, because $f_X(x) \neq f_{X|Y}(x|y)$.

(d) Find $P(X > Y)$. **Answer**:

$$\begin{aligned}
P(X > Y) &= \int_0^1 \int_x^1 (x+y)dydx \\
&= \int_0^1 \left[xy + \frac{1}{2}y^2\right]_x^1 dx \\
&= \int_0^1 \left[x + \frac{1}{2} - \left(x^2 + \frac{1}{2}x^2\right)\right] dx \\
&= \int_0^1 (-1.5x^2 + x + 0.5)dx \\
&= [-0.5x^3 + 0.5x^2 + 0.5x]_0^1 \\
&= 0.5.
\end{aligned}$$

4. Let X_1, X_2, X_3, X_4 be independent random variables, each having distribution $f(x) = \frac{8}{3}x^3 \exp(-2x)$ for $x \geq 0$.

(a) What kind of density is this? **Answer** $X_i \sim \text{gamma}(4, 2)$.

(b) Find the joint pdf of (X_1, X_2, X_3, X_4), namely $f(x_1, x_2, x_3, x_4)$. **Answer**: Because X_1, X_2, X_3, and X_4 are all mutually independent, the joint distribution $f(x_1, x_2, x_3, x_4)$ factors into the product of the four marginal distributions:

$$\begin{aligned}
f(x_1, x_2, x_3, x_4) &= f_{X_1}(x_1)f_{X_2}(x_2)f_{X_3}(x_3)f_{X_4}(x_4) \\
&= \left[\frac{8}{3}x_1^3e^{-2x_1}\right]\left[\frac{8}{3}x_2^3e^{-2x_2}\right]\left[\frac{8}{3}x_3^3e^{-2x_3}\right]\left[\frac{8}{3}x_4^3e^{-2x_4}\right] \\
&= \left[\frac{8}{3}\right]^4 [x_1x_2x_3x_4]^3e^{-2(x_1+x_2+x_3+x_4)}.
\end{aligned}$$

5. Let (X, Y) be jointly distributed with the pmf

x	y	$p(x,y)$
2	1	0.1
2	2	0.2
4	2	0.1
6	3	0.4
8	3	0.2

(a) Find $p_Y(y) = P(Y = y)$ and $p_X(x) = P(X = x)$. **Answer**: The marginal distribution of X is $P(X = 2) = 0.1 + 0.2 = 0.3$, $P(X = 4) = 0.1$, $P(X = 6) = 0.4$, $P(X = 8) = 0.2$. The marginal distribution of Y is $P(Y = 1) = 0.1$, $P(Y = 2) = 0.2 + 0.1 = 0.3$, $P(Y = 3) = 0.4 + 0.2 = 0.6$.

(b) What is $p_{X|Y}(x|2) = P(X = x|Y = 2)$ for $x \in \{2, 4, 6, 8\}$? **Answer**: First, $P(Y = 2) = p(2, 2) + p(4, 2) = 0.2 + 0.1 = 0.3$.

$$P(X = 2|Y = 2) = p(2, 2)/P(Y = 2) = \frac{0.2}{0.3} = \frac{2}{3}.$$

$$P(X = 4|Y = 2) = p(4, 2)/P(Y = 2) = \frac{0.1}{0.3} = \frac{1}{3}.$$

$$P(X = 6|Y = 2) = p(6, 2)/P(Y = 2) = \frac{0}{0.3} = 0.$$

$$P(X = 8|Y = 2) = p(8, 2)/P(Y = 2) = \frac{0}{0.3} = 0.$$

(c) What is $P(XY \leq 8)$? **Answer**: $P(XY \leq 8) = p(2, 1) + p(2, 2) + p(4, 2) = 0.4$.

(d) Are X and Y independent? **Answer**: No, $0 = P(X = 6|Y = 2) \neq P(X = 6) = 0.4$. Only need to find one example.

6. Use the pmf $p(x, y)$ for discrete (X, Y) in Chapter 3, Problem 1 (p. 107) to answer the following questions.

   (a) What is $P(X = Y^2)$? **Answer**: $P(X = Y^2) = p(1, 1)+p(4, 2) = 0.1+0.02 = 0.12$.

   (b) What is $P(X = 1)$? **Answer**: $P(X = 1) = p(1, 1) + p(1, 2) + p(1, 3) + p(1, 4) = 0.19$.

   (c) What is $P(X = 1|Y = 4)$? **Answer**: First need $P(Y = 4) = 0.18$. Then $P(X = 1|Y = 4) = P(X = 1, Y = 4)/P(Y = 4) = p(1, 4)/0.18 = 0.02/0.18 \approx 0.1111$.

   (d) Can X and Y be independent? **Answer**: No, $0.1111 \approx P(X = 1|Y = 4) \neq P(X = 1) = 0.19$.

   (e) Let $Z = X + Y$. What is the range of Z? What is $P(Z = 5)$? **Answer**: The range of Z is $\{2, 3, 4, 5, 6, 7, 8\}$. $P(Z = 5) = P(X + Y = 5) = p(2, 3) + p(1, 4) + p(4, 1) + p(3, 2) = 0.05 + 0.02 + 0.02 + 0.05 = 0.14$.

7. Use the joint density $f(x, y)$ for continuous (X, Y) in Chapter 3, Example A (p. 75, bottom) to answer the following questions.

   (a) Showing all work, verify that $f_Y(y) = \frac{12}{7}\left(\frac{1}{3} + \frac{y}{2}\right)$. **Answer**:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x, y)dx &= \int_0^1 \frac{12}{7}(x^2 + xy)dx \\
&= \frac{12}{7}\left[\frac{1}{3}x^3 + \frac{1}{2}x^2 y\right]_0^1 \\
&= \frac{12}{7}\left[\frac{1}{3}1^3 + \frac{1}{2}1^2 y - \left(\frac{1}{3}0^3 + \frac{1}{2}0^2 y\right)\right] \\
&= \frac{12}{7}\left(\frac{1}{3} + \frac{y}{2}\right).
\end{aligned}$$

   (b) Find $f_{X|Y}(x|y)$. **Answer**:

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{x^2 + xy}{\frac{1}{3} + \frac{y}{2}} \text{ on } 0 \leq y \leq 1.$$

   (c) From Example B (p. 76), $f_X(x) = \frac{12}{7}\left(x^2 + \frac{x}{2}\right)$. Does $f_X(x) = f_{X|Y}(x|y)$? That is, does X care about Y? **Answer**: $f_{X|Y}(x|y) \neq f_X(x)$, so X does care about Y.

   (d) Can X and Y be independent? **Answer**: No, the distribution of X changes with seeing different values $Y = y$.