Brief calculus review

Calculus begins with the notion of **limit**. We will briefly describe $\lim_{x \to a} f(x)$ for several types of functions on the blackboard, but only need the idea, not the definition, for much of calculus.

Differentiation

Recall that the **derivative** of a function f at the value x is defined to be

$$f'(x) = \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

if this limit exists. The value $f'(x)$ is the rate of change of f at the point x (see Figure 1).

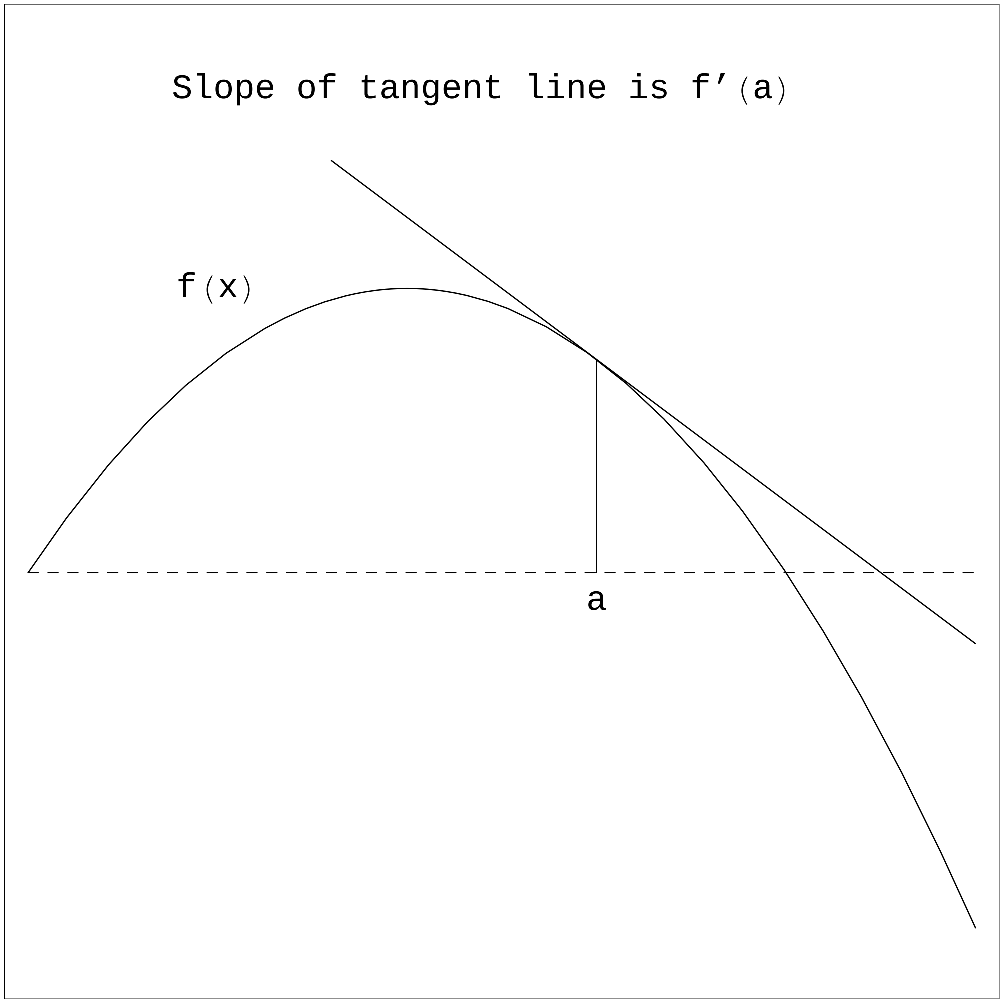

Figure 1: $f'(a)$ is the slope of the line tangent to $f(x)$ at a.

Some useful rules are

- $[f(x) + g(x)]' = f'(x) + g'(x)$.
- $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$.
- $[af(x)]' = af'(x)$ for any real a.
- $[x^a]' = ax^{a-1}$ for any real a. In particular, $[x]' = 1$.
- $[e^x]' = e^x$.
- $[f(g(x))]' = f'(g(x))g'(x)$. This one, called the "chain rule" is subtle.
- $[e^{-ax}]' = -ae^{-ax}$, which is implied by the last two. Using the chain rule, $f(x) = e^x$ and $g(x) = -ax$.

Examples

1.

$$\begin{aligned} [14x^2]' &= 14[x^2]' \\ &= 14[2x] \\ &= 28x. \end{aligned}$$

2.

$$\begin{aligned} \left[4e^{-4x} - \frac{0.5}{x}\right]' &= 4[e^{-4x}]' - 0.5[x^{-1}]' \\ &= 4(-4)e^{-4x} - 0.5(-1)x^{-2} \\ &= -16e^{-4x} + \frac{0.5}{x^2}. \end{aligned}$$

Integration

The integral of a function f over a and b is the area between the function f and the x-axis between a and b, and is denoted $\int_a^b f(x)dx$. The "dx" refers to which variable the function is being integrated with respect to. This matters more when we are integrating functions of two or more variables, e.g. $f(x, y)$. Integrals are important as they are how we obtain probabilities of a continuous random variable X being in an interval (or other set).

For example $P(a \leq X \leq b) = \int_a^b f(x)dx$ where $f(x)$ is the pdf for a continuous random variable X (see Figure 2).

Two useful rules are:

- $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ for real c.

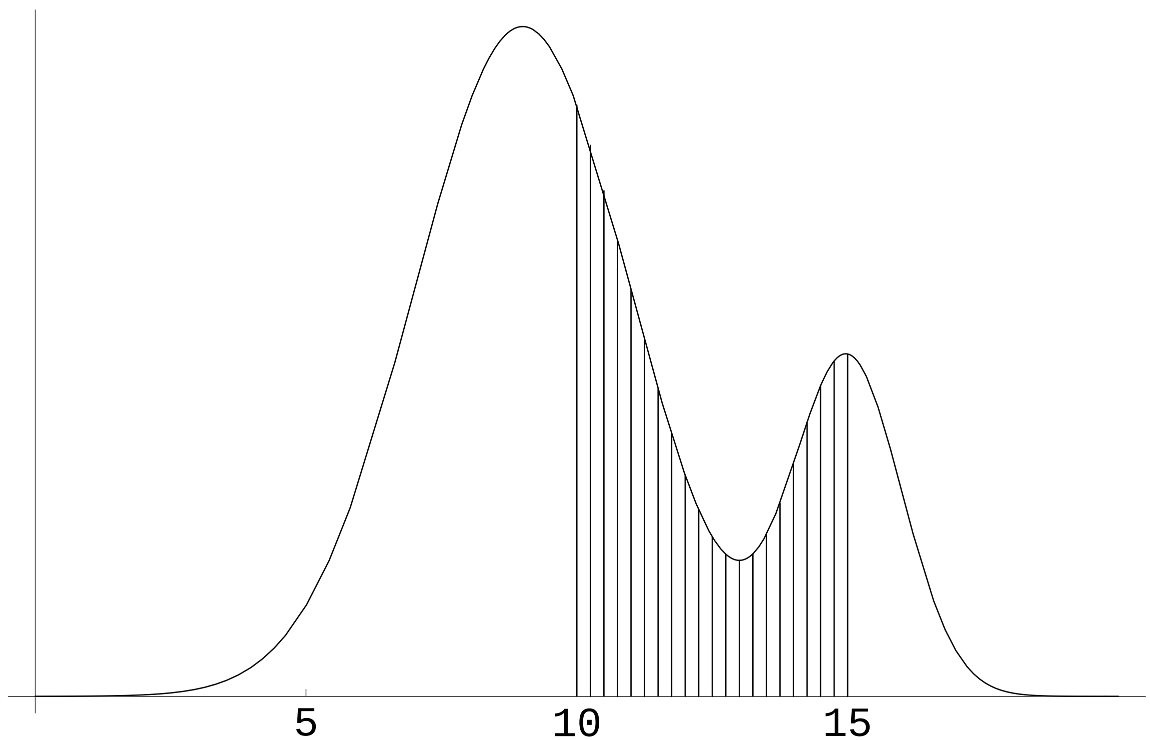

Figure 2: Probability density function $f(x)$ for one particular random variable X. Shaded region between 10 and 15 has area $\int_{10}^{15} f(x)dx = 0.3458$. That is, $P(10 \leq X < 15) = 0.3458$.

The *Fundamental Theorem of Calculus* (F.T.C.) ties together integration and differentiation. Say we want to evaluate $\int_a^b f(x)dx$. Then if we can find a $F(x)$ such that $F'(x) = f(x)$ then the F.T.C. tells us

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a).$$

This requires "differentiating in reverse" or asking the question "what function $F(x)$ would I have to differentiate to get $f(x)$?"

You might be able to evaluate some integrals using trial and error, or through careful thought. One thing you can always do is when you think you've found your $F(x)$, simply compute $F'(x)$ and make sure it's equal to $f(x)$. The definition of an integral is the limit of a particular sum approximating the area under the curve by rectangles. Don't worry if you don't remember it.

Example:

$$\begin{aligned}
\int_0^1 5x^2 dx &= 5\int_0^1 x^2 dx \\
&= 5\int_0^1 \left[\frac{1}{3}x^3\right]' dx \\
&= 5\left[\frac{1}{3}x^3\right]_0^1 \\
&= 5\left[\frac{1}{3}1^3 - \frac{1}{3}0^3\right] = \frac{5}{3}.
\end{aligned}$$

When one of the endpoints being integrated over is either $-\infty$ or ∞ then one "evaluates" $F(\infty)$ in terms of the *limiting value*. That is, for example, $[F(x)]_a^\infty = F(\infty) - F(a) = \lim_{x\to\infty} F(x) - F(a)$.

Note that

- $\lim_{x\to\infty} e^{-ax} = 0$ when $a \geq 0$.
- $\lim_{x\to\infty} x^{-a} = \lim_{x\to\infty} \frac{1}{x^a} = 0$ when $a > 0$.
- $\lim_{x\to\infty} x^a e^{-bx} = 0$ when $b > 0$ for any a.

$$
\begin{aligned}
\int_5^\infty \left[\frac{4}{x^3} - 2e^{-5x}\right] dx &= 4\int_5^\infty x^{-3} dx - 2\int_5^\infty e^{-5x} \\
&= 4\int_5^\infty \left[-\frac{1}{2}x^{-2}\right]' dx - 2\int_5^\infty \left[-\frac{1}{5}e^{-5x}\right]' dx \\
&= 4\left[-\frac{1}{2}x^{-2}\right]_5^\infty - 2\left[-\frac{1}{5}e^{-5x}\right]_5^\infty \\
&= 4\left[-\frac{1}{2}\lim_{x\to\infty} x^{-2} - -\frac{1}{2}5^{-2}\right] \\
&\quad -2\left[-\frac{1}{5}\lim_{x\to\infty} e^{-5x} - -\frac{1}{5}e^{-5(5)}\right] \\
&= 4\left[0 + \frac{1}{2}\frac{1}{25}\right] - 2\left[0 + \frac{1}{5}e^{-25}\right] \\
&= \frac{2}{25} - \frac{2}{5}e^{-25} = 0.08.
\end{aligned}
$$

There are many tools for evaluating integrals, mostly taught in Calc II, that we've probably forgotten. Examples are integration by parts, partial fractions, trigonometric identities, etc. The following uses integration by parts (but I hid it).

$$\begin{aligned}\int_0^\infty xe^{-x}dx &= \int_0^\infty [-(1+x)e^{-x}]'dx = [-(1+x)e^{-x}]_0^\infty \\ &= \lim_{x\to\infty} -(1+x)e^{-x} - -(1+0)e^{-0} \\ &= 1 - 0 = 1.\end{aligned}$$

This last limit is not trivial and requires the use of L' Hopital's rule from Calc I.

Note that $f(x) = xe^{-x}$ integrates to one and is non-negative over $R = [0, \infty)$, so it's a pdf for some random variable X. You should check in fact that $X \sim \text{gamma}(2, 1)$.

All integrals have a value, but not all integrals can be evaluated using tools from Calculus. For example, the function $f(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}$ is a perfectly valid function, and in fact it's a density called the *Gaussian Density*. However, you can't find a $F(x)$ such that $F'(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}$. Try it! However, integrals of $f(x)$ always exist, and in fact $\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-0.5x^2} dx = 0.5$.

Integrals that have no closed form are often said to be "intractable." They can be evaluated numerically, i.e. using special algorithms carried out by a computer. One easy approximation is to replace the integral by a sum which follows from the definition of integration:

$$\int_a^b f(x)dx \approx \frac{b-a}{N} \sum_{j=1}^{N} f\left(a + \frac{j}{N}(b-a)\right).$$

The accuracy of this approximation increases with N. For example, let's approximate $\int_{-1}^{2} \frac{1}{\sqrt{2\pi}} e^{-0.5x^2} dx = 0.818595$ to 6 decimal places.

N	approx
10	0.787769
100	0.815749
1,000	0.818312
10,000	0.818566
100,000	0.818592
1,000,000	0.818594

Partial derivatives

We will also consider functions of two or more variables. A function of two variables, generically denoted $f(x, y)$ takes any $(x, y) \in \mathbb{R}^2 = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\} = (-\infty, \infty) \times (-\infty, \infty)$ and returns a real number. Jointly defined continuous random variables (X, Y) have two-dimensional pdf's $f(x, y) \geq 0$.

The *partial derivative* of any two-dimensional function $\frac{\partial}{\partial x} F(x, y)$ gives the rate of change of F in the x-direction at a fixed (x, y). The partial derivative $\frac{\partial}{\partial y} F(x, y)$ gives the rate of change of $F(x, y)$ in the y direction at a given (x, y). The second partial derivative $\frac{\partial^2}{\partial x \partial y} F(x, y)$ is defined to be $\frac{\partial}{\partial x} \left[\frac{\partial}{\partial y} f(x, y) \right]$. Partial derivatives are evaluated *exactly the same way as ordinary derivatives* except that the other variables are treated as constants.

Examples

1.

$$\begin{aligned}
\frac{\partial}{\partial y}\left[2y^2e^x - 5e^{-xy}\right] &= 2e^x\frac{\partial}{\partial y}y^2 - 5\frac{\partial}{\partial y}e^{-xy} \\
&= 2e^x\,[2y] - 5[-xe^{-xy}] \\
&= 4e^x y + 5xe^{-xy}
\end{aligned}$$

2.

$$\begin{aligned}
\frac{\partial^2}{\partial x \partial y}\left[\frac{1}{4}x^2y^2\right] &= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\frac{1}{4}x^2y^2\right] \\
&= \frac{\partial}{\partial x}\left[\frac{1}{2}x^2y\right] \\
&= xy.
\end{aligned}$$

Multiple Integrals

For now, we only consider integrating functions of two variables $f(x, y)$. The multiple integral of a function f over a region A in the plane ($A \subset \mathbb{R}^2$), is the volume of the region between the x-y plane and f over A. This integral is written

$$\iint_A f(x, y) dx dy.$$

One way to evaluate such integrals uses a theorem from Calculus II. Basically, you need to write the set A as
$A = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$ or as
$A = \{(x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$ for some functions (either g_1 and g_2 or h_1 and h_2).

If you can do this, then the integral is evaluated as two regular integrals *in succession* given by either of the following:

$$\begin{aligned}
\iint_A f(x,y)dxdy &= \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)dy\right] dx \\
&= \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)dx\right] dy.
\end{aligned}$$

For example, say we want to integrate $f(x, y) = x^2y$ over the unit disk given by $A = \{(x, y) : x^2 + y^2 \leq 1\}$. We can write A as $A = \{(x, y) : -1 \leq x \leq 1, -\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}\}$ and so

$$\iint_A f(x, y)dxdy = \int_{-1}^{1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2ydy \right] dx$$

Let's do this one:

We have

$$\begin{aligned}
\iint_A x^2 y dx dy &= \int_{-1}^{1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 y dy \right] dx \\
&= \int_{-1}^{1} \left[x^2 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{\partial}{\partial y} \frac{1}{2} y^2 dy \right] dx \\
&= \int_{-1}^{1} \left[x \left[\frac{1}{2} y^2 \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \right] dx \\
&= \int_{-1}^{1} 0 dx = 0.
\end{aligned}$$

Let's do one more example. Soon we will define the joint pdf for a random pair (X, Y). For now, consider the function $f(x, y)$ given by

$$f(x,y) = \left\{ \begin{array}{ll} 2 & \text{for } 0 \le x \le 1 \text{ and } x \le y \le 1 \\ 0 & \text{otherwise} \end{array} \right\}.$$

Say we want $P(X \le 0.5)$. The *range* of (X, Y), $R = \{(x, y) : f(x, y) > 0\}$, is explicitly given in the definition of $f(x, y)$ above. So when integrating $f(x, y)$ over $A = \{(x, y) : x \le 0.5\}$ to get the desired probability, we only need to consider that part of A that is also in R. $A \cap R = \{(x, y) : 0 \le x \le 0.5, x \le y \le 1\}$ and so

$$
\begin{aligned}
\iint_A f(x,y)\,dx\,dy &= \int_0^{0.5}\left[\int_x^1 2dy\right]dx \\
&= \int_0^{0.5} [2y]_x^1 dx \\
&= \int_0^{0.5} (2-2x)dx \\
&= [2x - x^2]_0^{0.5} \\
&= 1 - 1/4 = 0.75.
\end{aligned}
$$

That is, $P(X \leq 0.5) = 0.75$. We'll discuss another route to obtaining this probability via the *marginal* distribution function of X later on.