Total 30p

Homework 2
Answer Key

Chapter 2

(2) a)

Number of heads before the first tail	0	1	2	3	4
pmf	1/2	1/4	1/8	1/16	1/16
cdf	1/2	3/4	7/8	15/16	1

$S = \{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT,$
$THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT\}$

2p $\dim(S) = 16 = 2^4$

$P(X=0) = \frac{8}{16} = \frac{1}{2}$; $P(X=1) = \frac{4}{16} = \frac{1}{4}$

$P(X=2) = \frac{2}{16} = \frac{1}{8}$; $P(X=3) = \frac{1}{16}$; $P(X=4) = \frac{1}{16}$

$P(X \leq 1) = \frac{12}{16} = \frac{3}{4}$; $P(X \leq 2) = \frac{12}{16} + \frac{2}{16} = \frac{14}{16} = \frac{7}{8}$

b)

Number of heads following the first tail	0	1	2	3
pmf	5/16	6/16	4/16	1/16
CDF	5/16	11/16	15/16	1

2p

y = nr. of heads
$\Rightarrow 4-y$ = nr. of tails
$\Rightarrow 2y-4$ = the diff

c)

Nr. of heads − nr. of tails	−4	−2	0	2	4
pmf	1/16	4/16	6/16	4/16	1/16
CDF	1/16	5/16	11/16	15/16	1

2p

d)

Nr. of tails * nr. of heads $y(4-y)$	0	3	4
pmf	2/16	8/16	6/16
CDF	2/16	10/16	1

2p

(12) In both cases we are dealing with the binomial distribution, $p=\frac{1}{2}$

9 heads in 10 tosses $\Rightarrow \binom{10}{9}\left(\frac{1}{2}\right)^{9}\left(\frac{1}{2}\right)^{10-9} = 10\cdot\left(\frac{1}{2}\right)^{10} = \frac{10}{1024} = 0.00977$

2p 18 heads in 20 tosses $\Rightarrow \binom{20}{18}\cdot\left(\frac{1}{2}\right)^{18}\left(\frac{1}{2}\right)^{20-18} = \frac{\cancel{20}^{10}\cdot 19}{\cancel{2}}\cdot\frac{1}{2^{20}} = \frac{190}{1048576} = 0.00018$

Therefore, 9 h in 10 t is more likely than 18 h in 20 t.

(14) Let X be the random variable which denotes the number of attempts

a) X can take any values $1, 2, \dots, +\infty$

$P(X=1) = P(A \text{ makes the basket in the first shot}) = p_1$

$P(X=2) = P(A \text{ misses, } B \text{ hits it from the first}) \overset{\text{ind.}}{=} (1-p_1)\cdot p_2$

2p

$P(X=2n) = P(A \text{ misses } n \text{ shots, } B \text{ misses } (n-1) \text{ shots but then hits})$
$\overset{\text{ind}}{=} (1-p_1)^n(1-p_2)^{n-1}p_2$

$P(X=2n+1) = P(A \text{ and } B \text{ miss } n \text{ shots, then } A \text{ hits})$
$= (1-p_1)^n(1-p_2)^n p_1$

b) $P(A \text{ wins}) = P(X=1) + P(X=3) + \cdots + P(X=2n+1) + \cdots$

$$= \sum_{i \geq 0} P(X=2i+1) = \sum_{i \geq 0} p_1 (1-p_1)^i (1-p_2)^i = p_1 \frac{1}{1-(1-p_1)(1-p_2)} \qquad 2p$$

$$= \frac{p_1}{p_1 + p_2 - p_1 p_2}$$

(20) $X \sim Geo(0.5)$

By definition,

$$P(X \le k) = \sum_{i=1}^{k} p\cdot(1-p)^{i-1} = \underbrace{\sum_{i\ge1} p\cdot q^{i-1}}_{=1} - \sum_{j\ge k+1} p\cdot q^{j-1}$$

3p $$\sum_{j\ge k+1} p q^{j-1} = \sum_{\substack{j\ge1 \\ j\to j+k}} p\cdot q^{j+k-1} = \frac{p\cdot q^k}{q}\sum_{j\ge1} q^j = \frac{p\cdot q^k}{\cancel{q}}\cdot\frac{\cancel{q}}{\underbrace{1-q}_{=p}} = q^k$$

$$\Rightarrow P(X\le k) = 1-q^k = 1-(0.5)^k \approx 0.99$$

$$\Rightarrow (0.5)^k \approx 0.01$$

$$\Rightarrow k \approx \log_{0.5}(0.01) = \frac{\ln(0.01)}{\ln(0.5)} = \frac{-4.605}{-0.693} = 6.645$$

(22) P(thrown more than 3 times) = 1 − P(thrown once) − − P(thrown twice) − P(thrown 3 times)

$$P(\text{thrown once}) = P(HHH \text{ or } TTT) \overset{ind}{=} \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^2$$

P(thrown twice) = P(first miss AND second wright)

$$= (1-P(\text{first wright}))\cdot P(\text{second wright}) = \left(1-\frac{1}{4}\right)\cdot\frac{1}{4} = \frac{3}{16}$$

2p P(thrown 3 times) = P(first miss AND second miss AND third right)

$$= \left(1-\frac{1}{4}\right)\left(1-\frac{1}{4}\right)\cdot\frac{1}{4} = \frac{9}{64}$$

$$\Rightarrow P(\text{thrown more than 3 times}) = 1 - \frac{1}{4} - \frac{3}{16} - \frac{9}{64} = \frac{27}{64} = 0.4219$$

(26) Let X = # of times that the mathematician is trapped in the elevator

$\Rightarrow X \sim Bin\left(n = 5 \cdot 52 \cdot 10 = 2600 \text{ days}, \; p = \frac{1}{10000} = 0.0001\right)$

Since n is large and p is small $\Rightarrow Bin(n,p) \approx Poi(\lambda = n \cdot p = 0.26)$

$$\Rightarrow P(X=0) = \frac{(0.26)^0 \cdot e^{-0.26}}{0!} = 0.7711$$

2p

$$P(X=1) = \frac{(0.26)^1 \cdot e^{-0.26}}{1!} = 0.2005$$

$$P(X=2) = \frac{(0.26)^2 \cdot e^{-0.26}}{2!} = 0.0261$$

(34) $f(x) = \begin{cases} \frac{1+\alpha x}{2} & , \text{if } x \in [-1,1] \\ 0 & , \text{o.w.} \end{cases}$, where $\alpha \in [-1,1]$

Since α and $x \in [-1,1] \Rightarrow \alpha \cdot x \in [-1,1] \Rightarrow 1 + \alpha x \geq 0$

$\Rightarrow f(x) \geq 0 \quad (1)$,

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-1}^{1} \frac{1+\alpha x}{2}\,dx = \frac{1}{2}\left(x + \alpha \cdot \frac{x^2}{2}\right)\Big|_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(1+\frac{\alpha}{2}\right) - \left(-1+\frac{\alpha}{2}\right)\right] = \frac{2}{2} = 1 \Rightarrow \int_{-\infty}^{+\infty} f(x)\,dx = 1 \quad (2)$$

3p

$\Rightarrow$ from (1) and (2) we conclude that $f(x)$ is a pdf

$$F(x) = \int_{-\infty}^{x} f(y)\,dy \overset{1 \geq x \geq -1}{=\!=\!=} \int_{-1}^{x} \frac{1+\alpha y}{2}\,dy = \frac{1}{2}\left(x + \frac{\alpha x^2}{2} + 1 - \frac{\alpha}{2}\right)$$

$$\Rightarrow F(x) = \begin{cases} 0 & , \text{if } x < -1 \\ \frac{1}{2}\left(\frac{\alpha x^2}{2} + x + 1 - \frac{\alpha}{2}\right) & , \text{if } x \in [-1,1] \\ 1 & , x > 1 \end{cases}$$ is the corresponding CDF

The p-th quartile is X_p st $F(X_p)=p$, ie

$$\alpha X_p^2 + 2X_p + 2 - \alpha - 4p = 0$$

$$\Rightarrow \quad X_p = \frac{-1 + \sqrt{1-\alpha(2-\alpha-4p)}}{\alpha}$$

The median is the 0.5^{th} quartile

$$\Rightarrow \quad X_{0.5} = \frac{-1 + \sqrt{1-\alpha(2-\alpha-2)}}{\alpha} = \frac{-1+\sqrt{1+\alpha^2}}{\alpha}$$

(40)

$$f(x) = \begin{cases} cx^2, & \text{if } 0 \le x \le 1 \\ 0, & \text{ow} \end{cases}$$

1p a) $1 = \int_{-\infty}^{+\infty} f(x)\,dx = c \cdot \int_0^1 x^2\,dx = c \cdot \frac{x^3}{3}\Big|_0^1 = \frac{c}{3} \Rightarrow c = 3$

b) $\int_0^x cy^2\,dy = c \cdot \frac{y^3}{3}\Big|_0^x = x^3$

1p $\Rightarrow F(x) = \begin{cases} 0, & \text{if } x < 0 \\ x^3, & \text{if } x \in [0,1] \\ 1, & \text{if } x > 1 \end{cases}$

2p c) $P(0.1 \le X < 0.5) = P(X < 0.5) - P(X < 0.1)$

$= F(0.5) - F(0.1) = 0.5^3 - 0.1^3 = 0.124$

(46) $T \sim Exp(\lambda) \Rightarrow F(t) = \begin{cases} 1 - e^{-\lambda t} & , t \geq 0 \\ 0 & , t < 0 \end{cases}$

2p We are given that $F(1) = 0.05$

$\Rightarrow e^{-\lambda} = 0.95 \Rightarrow -\lambda = \ln 0.95 \Rightarrow \lambda = 0.0513$