Supplement to "Statistical methods research done as science rather than mathematics"

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The predictor of $\hat{\rho} = -1$, reproduced from the main paper's equation (10), is

$$\left(\frac{Ns - N - 1}{(1 + r/s)(1 + r/q)}\right)\left[1 - \left(\frac{Ns - 2}{Ns - N - 1}\right)\frac{1 - \frac{N-1}{N(s-2)}\rho}{1 + \frac{2(N-1)}{N(s-2)}\frac{(1+r/s)(1+r/q)+\rho}{(1+r/s)(1+r/q)-1}}\right]. \tag{1}$$

1 The predictor of $\hat{\rho} = -1$ is positive

In the expression for the predictor in equation (1) above,

$$\begin{aligned}
\frac{1 - \frac{N-1}{N(s-2)}\rho}{1 + \frac{2(N-1)}{N(s-2)}\frac{(1+r/s)(1+r/q)+\rho}{(1+r/s)(1+r/q)-1}} &= \frac{N(s-2) - (N-1)\rho}{N(s-2) + 2(N-1)\frac{(1+r/s)(1+r/q)+\rho}{(1+r/s)(1+r/q)-1}} \\
&< \frac{N(s-2) + (N-1)}{N(s-2) + 2(N-1)} = \frac{Ns - N - 1}{Ns - 2}
\end{aligned} \tag{2}$$

for all finite positive N, s, and r and $\rho \in (-1, 1)$. It follows immediately that the predictor is positive for these N, s, r and ρ.

2 The predictor of $\hat{\rho} = +1$

The predictor for $\hat{\rho} = +1$ is derived by the same sequence of steps used to derive the predictor for $\hat{\rho} = -1$, which gives

$$-\left(\frac{Ns - N - 1}{(1 + r/s)(1 + r/q)}\right)\left[1 - \left(\frac{Ns - 2}{Ns - N - 1}\right)\frac{1 + \frac{N-1}{N(s-2)}\rho}{1 + \frac{2(N-1)}{N(s-2)}\frac{(1+r/s)(1+r/q)-\rho}{(1+r/s)(1+r/q)-1}}\right]. \tag{3}$$

Comparison to equation (1) above shows that equation (3) above differs because of three sign changes: a minus sign at the far left of equation (3) and changes in the signs of the summands including ρ in the numerator and denominator of the complicated fraction at the far right of equation (3).

In the main paper's tables, to make this predictor more similar to the predictor for $\hat{\rho} = -1$, the minus sign at the far left of equation (3) has been omitted.

3 Three bullets early in the main paper's Section 3

The three bullets are:

- Given N, s, and ρ, as r increases — i.e., as the error variance σ_e^2 increases relative to the random-effect variance σ_r^2 — the predictor goes to zero.
- Given ρ and r, as either N or s increases, the predictor goes to infinity.
- Given N, s, and r, as ρ goes to -1, the predictor goes to zero.

3.1 Increasing r

In equation (1) above, as r increases,

$$\frac{(1+r/s)(1+r/q)+\rho}{(1+r/s)(1+r/q)-1} \to 1, \tag{4}$$

so the expression in square brackets goes to a finite positive number for given legal values of N, s, and ρ. However, as r increases, the expression in round brackets at the far left of equation (1) goes to 0, so the predictor goes to zero.

3.2 Increasing N

In equation (1) above, inside the square brackets are three fractions involving N, and as N increases, each such fraction goes to a finite, positive number:

$$\frac{Ns-2}{Ns-N-1} \to \frac{s}{s-1}; \quad \frac{N-1}{N(s-2)} \to \frac{1}{s-2}; \quad \frac{2(N-1)}{N(s-2)} \to \frac{2}{s-2}. \tag{5}$$

Therefore, the expression in square brackets goes to a finite, positive number. However, as N increases, the expression in round brackets at the far left of equation (1) increases without bound, so the predictor does as well.

3.3 Increasing s

Because $s = 2m+1$ and $q = (2m^2+3m+1)/3m$, as s increases, so does q. As s increases,

$$N(s-2)[(1+r/s)(1+r/q)-1] = \frac{N(s-2)}{s}[(s+r)(1+r/q)-s] \to Nr, \tag{6}$$

so the expression in square brackets in equation (1) goes to a finite, positive number. However, as s increases, the expression in round brackets at the far left of equation (1) increases without bound, so the predictor does as well.

3.4 Letting ρ go to -1

As $\rho \to -1$,

$$\frac{1 - \frac{N-1}{N(s-2)}\rho}{1 + \frac{2(N-1)}{N(s-2)} \frac{(1+r/s)(1+r/q)+\rho}{(1+r/s)(1+r/q)-1}} \to \frac{1 + \frac{N-1}{N(s-2)}}{1 + \frac{2(N-1)}{N(s-2)}} = \frac{Ns - N - 1}{Ns - 2}. \tag{7}$$

Thus the expression in square brackets in equation (1) goes to zero and so does the predictor.

4 Section 5's results for $\hat{\rho} = \pm 1$ and $\hat{\rho} =$ NaN

Figures 1 and 2 below are analogous to the main paper's Figure 2, showing the percent of simulated datasets giving $\hat{\rho} = \pm 1$ and $\hat{\rho} =$ NaN respectively as a function of r and each of the other factors. Note that the vertical scales in these plots are 0 to 60% and 0 to 40%, while Figure 2's vertical scale is 0 to 100%. The Monte Carlo standard error for each plotted percent is less than 0.8 percentage points.

Figure 1: Percent datasets giving $\hat{\rho} = \pm 1$ as a function of $\log_{10} r$.

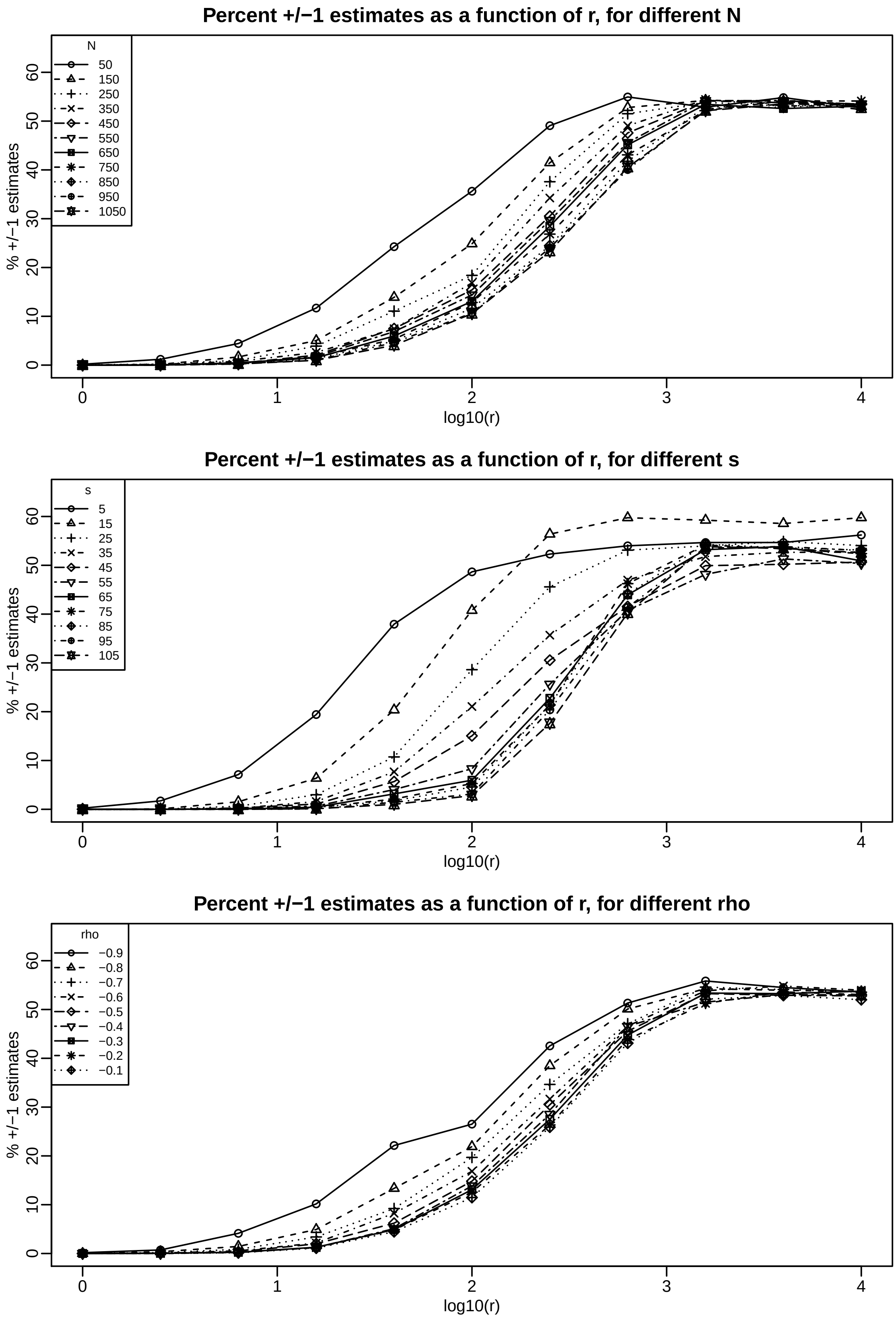

Figure 2: Percent datasets giving $\hat{\rho} = \text{NaN}$ as a function of $\log_{10} r$.