Models more general than two-variance models

It would be great if this analysis could be done for all mixed linear models.

Alas, it can't.

I'll show this for 2NRCAR models.

H2013 shows it can be done for some big classes of models.

- I'll list these and spare you the details, except for one.

Two expedients may help for models with RLs that can't be simplified.

- Each involves ignoring some part of the RL.
- This makes them approximate or sloppy, as you prefer.
- They appear to provide useful information and might suggest ways to extend or supersede the approach used for two-variance models.

You can't always diagonalize the RL for 2NRCAR models

Suppose $\mathbf{y} = \boldsymbol{\delta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim N_n(\mathbf{0}, \sigma_e^2 \mathbf{I}_n)$, and $\boldsymbol{\delta}$ has a 2NRCAR density:

$$f(\boldsymbol{\delta}|\sigma_{s1}^2, \sigma_{s2}^2) \propto \exp\left[-\frac{1}{2}\boldsymbol{\delta}'\left(\mathbf{Q}_1/\sigma_{s1}^2 + \mathbf{Q}_2/\sigma_{s2}^2\right)\boldsymbol{\delta}\right];$$

- $\mathbf{Q}_k = (q_{ij,k}), k = 1, 2$ encodes class-k neighbor pairs
- σ_{sk}^2 controls similarity induced by class-k neighbor pairs.

From Newcomb (1961) & Graybill (1983, Theorem 12.2.12):

- $\exists$ non-singular $\mathbf{B} \ni \mathbf{Q}_k = \mathbf{B}'\mathbf{D}_k\mathbf{B}$, where
- $\mathbf{D}_k$ is diagonal with non-negative diagonal elements.
- AND $\mathbf{B}$ is orthogonal $\Leftrightarrow$ $\mathbf{Q}_1\mathbf{Q}_2$ is symmetric.

$\mathbf{y} = \boldsymbol{\delta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim N_n(\mathbf{0}, \sigma_e^2 \mathbf{I}_n)$ and $\boldsymbol{\delta}$ has a 2NRCAR density:

$$f(\boldsymbol{\delta}|\sigma_{s1}^2, \sigma_{s2}^2) \propto \exp\left[-\frac{1}{2}\boldsymbol{\delta}'\left(\mathbf{Q}_1/\sigma_{s1}^2 + \mathbf{Q}_2/\sigma_{s2}^2\right)\boldsymbol{\delta}\right];$$

$\exists$ non-singular $\mathbf{B} \ni \mathbf{Q}_k = \mathbf{B}'\mathbf{D}_k\mathbf{B}$

- $\mathbf{D}_k$ has I_k zero diagonal elements; $I_k = \#\mathbf{Q}_k$'s zero eigenvalues.
- $I_k \geq I = \#(\mathbf{Q}_1 + \mathbf{Q}_2)$'s zero eigenvalues $= \#$ zero diagonal entries in $\mathbf{D}_1 + \mathbf{D}_2$.

Define $\mathbf{D}_{k+}$ = upper-left $(n - I) \times (n - I)$ submatrix of $\mathbf{D}_k$

Let $d_{kj}, j = 1, \ldots, n - I$, be the diagonal elements of $\mathbf{D}_{k+}$.

Finally, define the precisions $\tau_e = 1/\sigma_e^2$, $\tau_1 = 1/\sigma_{s1}^2$, and $\tau_2 = 1/\sigma_{s2}^2$.

Then the joint posterior for $(\boldsymbol{\delta}, \tau_e, \tau_1, \tau_2) \propto$

$$\pi(\tau_e, \tau_1, \tau_2) \qquad |\tau_e \mathbf{I}_n|^{0.5} \quad \exp\left(-0.5\tau_e(\mathbf{y}-\boldsymbol{\delta})'(\mathbf{y}-\boldsymbol{\delta})\right)$$
$$\prod_{j=1}^{n-I} (d_{1j}\tau_1 + d_{2j}\tau_2)^{0.5} \exp\left(-0.5\boldsymbol{\delta}'(\tau_1\mathbf{Q}_1 + \tau_2\mathbf{Q}_2)\boldsymbol{\delta}\right).$$

$\prod_{j=1}^{n-I} (d_{1j}\tau_1 + d_{2j}\tau_2)$ is the determinant of $\tau_1 \mathbf{D}_{1+} + \tau_2 \mathbf{D}_{2+}$

Integrate out $\boldsymbol{\delta}$ to give the RL $\propto$:

$$|\tau_e \mathbf{I}_n|^{0.5} \quad \prod_{j=1}^{n-I} (d_{1j}\tau_1 + d_{2j}\tau_2)^{0.5} \quad |\mathbf{H}|^{-0.5} \exp\left(-0.5\tau_e(\mathbf{y}'\mathbf{y} - \tau_e\mathbf{y}'\mathbf{H}^{-1}\mathbf{y})\right)$$

where $\mathbf{H} = \tau_e\mathbf{I}_n + \tau_1\mathbf{Q}_1 + \tau_2\mathbf{Q}_2$. <u>We need to diagonalize $\mathbf{H}$</u>.

We need to diagonalize

$$\begin{aligned}\mathbf{H} &= \tau_e \mathbf{I}_n + \tau_1 \mathbf{Q}_1 + \tau_2 \mathbf{Q}_2 \\ &= \tau_e \mathbf{I}_n + \mathbf{B}'(\tau_1 \mathbf{D}_1 + \tau_2 \mathbf{D}_2)\mathbf{B} \\ &= \mathbf{B}' \left[\tau_e (\mathbf{B}\mathbf{B}')^{-1} + \tau_1 \mathbf{D}_1 + \tau_2 \mathbf{D}_2\right] \mathbf{B}\end{aligned}$$

It's enough to diagonalize the part inside the square brackets.

This happens $\Leftrightarrow$ $\mathbf{B}$ is $\perp$ $\Leftrightarrow$ $\mathbf{Q}_1\mathbf{Q}_2$ is symmetric; not true in general.

The RL for the 2NRCAR model diagonalizes $\Leftrightarrow$ $\mathbf{Q}_1\mathbf{Q}_2$ is symmetric.

- Examples: Row-and-column grids; spatiotemporal 2NRCAR.
- Counterexamples: Periodontal 2NRCARs.

The RL does diagonalize for some big classes of models

Balanced designs:

- For any design that satisfies the conditions of general balance and is also an orthogonal design, the RL diagonalizes.
- This includes everything you think of as a balanced ANOVA.

Separable models:

- Model $\mathbf{y} = \boldsymbol{\delta} + \boldsymbol{\epsilon}, \qquad \boldsymbol{\epsilon} \sim N_n(\mathbf{0}, \sigma_e^2 \mathbf{I}_n)$
- $\boldsymbol{\delta} \sim$ normal mean $\mathbf{0}$, precision $\sum_{k=1}^{K} \tau_k \mathbf{Q}_k$ for $\mathbf{Q}_k$ $n \times n$, τ_k scalar.
- $\mathbf{Q}_k = \mathbf{A}_1 \otimes \cdots \otimes \mathbf{A}_M$ with $\mathbf{A}_l = \mathbf{I}$ for $l \neq k$, $\mathbf{A}_k$ is psd.
- This includes diagonalizable kNRCAR models, among others.
- This isn't general balance; RLs are not gamma GLM likelihoods.

Miscellaneous others:

- Clustering + heterogeneity model.
- Certain extensions of separable models.

Two expedients for problems that don't diagonalize

Expedient #1: Ignore the error variance

- Example: 2NRCAR puzzle.
- Brian Reich did this work *before* developing the re-expressed RL; this inspired the re-expression.
- I'll show you this.

Expedient #2: Ignore the non-zero off-diagonals

- Example: Optical imaging (DLM) puzzle.
- I won't show you this – it's in H2013, Sec. 17.2.2.

Expedient #1: Ignore the error variance – 2NRCAR models

$\mathbf{y} = \boldsymbol{\delta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim N_n(\mathbf{0}, \sigma_e^2 \mathbf{I}_n)$ and $\boldsymbol{\delta}$ has a 2NRCAR density:

$$f(\boldsymbol{\delta}|\sigma_{s1}^2, \sigma_{s2}^2) \propto \exp\left[-\frac{1}{2}\boldsymbol{\delta}'\left(\mathbf{Q}_1/\sigma_{s1}^2 + \mathbf{Q}_2/\sigma_{s2}^2\right)\boldsymbol{\delta}\right];$$

$\exists$ non-singular $\mathbf{B} \ni \mathbf{Q}_k = \mathbf{B}'\mathbf{D}_k\mathbf{B}$

- $\mathbf{D}_k$ has I_k zero diagonal elements; $I_k = \#\mathbf{Q}_k$'s zero eigenvalues.
- $I_k \geq I = \#(\mathbf{Q}_1 + \mathbf{Q}_2)$'s zero eigenvalues $= \#$ zero diagonal entries in $\mathbf{D}_1 + \mathbf{D}_2$.

Let d_{kj} be $\mathbf{D}_k$'s diagonals, for $j \ni d_{1j} > 0$ <u>or</u> $d_{2j} > 0$, $j = 1, \ldots, n - I$.

Finally, define the precisions $\tau_e = 1/\sigma_e^2$, $\tau_1 = 1/\sigma_{s1}^2$, and $\tau_2 = 1/\sigma_{s2}^2$.

$\mathbf{y} = \boldsymbol{\delta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim N_n(\mathbf{0}, \sigma_e^2 \mathbf{I}_n)$ and $\boldsymbol{\delta}$ has a 2NRCAR density

Then the joint posterior for $(\boldsymbol{\delta}, \tau_e, \tau_1, \tau_2) \propto$

$$\pi(\tau_e, \tau_1, \tau_2) \qquad |\tau_e \mathbf{I}_n|^{0.5} \quad \exp\left(-0.5\tau_e(\mathbf{y}-\boldsymbol{\delta})'(\mathbf{y}-\boldsymbol{\delta})\right)$$
$$\prod_{j=1}^{n-I} (d_{1j}\tau_1 + d_{2j}\tau_2)^{0.5} \exp\left(-0.5\boldsymbol{\delta}'(\tau_1\mathbf{Q}_1 + \tau_2\mathbf{Q}_2)\boldsymbol{\delta}\right).$$

Expedient #1: Condition on $(\tau_e, \boldsymbol{\delta})$ and ignore $\pi(\tau_e, \tau_1, \tau_2)$, so

$$\begin{aligned}
\pi(\tau_1, \tau_2 | \boldsymbol{\delta}) &\propto \prod_{j=1}^{n-I} (d_{1j}\tau_1 + d_{2j}\tau_2)^{0.5} \exp\left(-0.5\boldsymbol{\delta}'\mathbf{B}'(\tau_1\mathbf{D}_1 + \tau_2\mathbf{D}_2)\mathbf{B}\boldsymbol{\delta}\right) \\
&= \prod_{j=1}^{n-I} (d_{1j}\tau_1 + d_{2j}\tau_2)^{0.5} \exp\left(-0.5\sum_{j=1}^{n-I} u_j^2 (d_{1j}\tau_1 + d_{2j}\tau_2)\right),
\end{aligned}$$

for $\mathbf{u} = \mathbf{B}\delta$.

$$\pi(\tau_1, \tau_2|\boldsymbol{\delta}) \propto \prod_{j=1}^{n-l} (d_{1j}\tau_1 + d_{2j}\tau_2)^{0.5} \exp\left(-0.5 \sum_{j=1}^{n-l} u_j^2 (d_{1j}\tau_1 + d_{2j}\tau_2)\right)$$

for $\mathbf{u} = \mathbf{B}\boldsymbol{\delta}$; $\mathbf{B}$ is known, we're conditioning on the unknown $\boldsymbol{\delta}$.

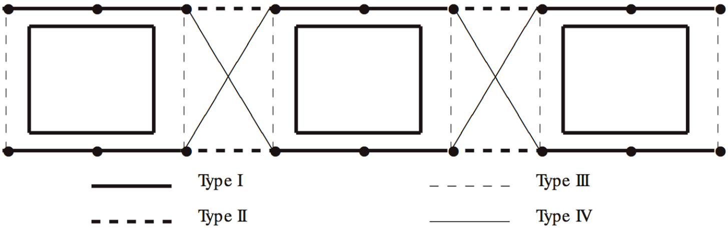
This looks like the re-expressed RL for a 2-variance model, but:

- In a 2-variance model, σ_e^2 can have free terms but σ_s^2 can't.
- Here, however, τ_1 and τ_2 are symmetric.
- So consider free and mixed terms for both classes of neighbor pairs.

Definitions:

- **u**-free terms for τ_1 have $d_{2j} = 0$; $\exists\ I_2 - I$ of them.
- **u**-free terms for τ_2 have $d_{1j} = 0$, $\exists\ I_1 - I$ of them.
- **u**-mixed terms have $d_{1j} > 0$ and $d_{2j} > 0$; $\exists\ n - I_1 - I_2 + I$ of them.

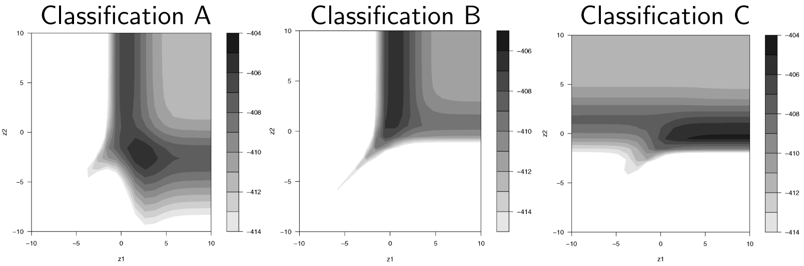
Recall: Periodontal neighbor pairs are of four distinct types.



Consider three ways of defining 2 classes of neighbor pairs:

Classification	Type of nbr pair I	II	III	IV	Description
A	1	1	2	2	Sides vs. interproximal
B	1	2	2	2	Direct vs. interproximal
C	2	1	2	2	Type II vs. others

Contours of one person's marginal posterior of (z_1, z_2), $z_k = \log(\tau_k/\tau_1)$.



The "legs" arise from **u**-free terms (intuition on the next slide).

					Counts of		
		# of islands			**u**-free terms		**u**-mixed
Classification	n	I	I_1	I_2	for z_1	for z_2	terms
A	162	3	6	84	81	3	75
B	162	3	54	84	81	51	27
C	162	3	114	3	0	111	48

Reich et al (2007, Section 4) presents the argument.

Intuition:

- The **u**-free terms for z_1 considered alone have contours parallel to the z_2 axis. This gives the vertical "leg".
- The **u**-free terms for z_2 considered alone have contours parallel to the z_1 axis. This gives the horizontal "leg".
- Classification C's contour plot has only one "leg" parallel to the z_1 axis because it has no **u**-free terms for z_1.

So why don't the legs go down & to the left? Explained in H2013, p. 385.