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while if $\sigma_C{}^2$ is large, $E(C)$ will exceed the right-hand side. A test criterion is

$$F' = \{(C) + (ABC)\}/\{(AC) + (BC)\}$$

where (C) denotes the mean square for C, and so on. The approximate degrees of freedom are

$$n_1 = \frac{\{(C) + (ABC)\}^2}{\dfrac{(C)^2}{f_C} + \dfrac{(ABC)^2}{f_{ABC}}}$$

$$n_2 = \frac{\{(AC) + (BC)\}^2}{\dfrac{(AC)^2}{f_{AC}} + \dfrac{(BC)^2}{f_{BC}}}$$

12.12—The split-plot or nested design. It is often desirable to get precise information on one factor and on the interaction of this factor with a second, but to forego such precision on the second factor. For example, three sources of vitamin might be compared by trying them on three males of the same litter, replicating the experiment on 20 litters. This would be a randomized blocks design with high precision, providing 38 degrees of freedom for error. Superimposed on this could be some experiment with the litters as units. Four types of housing could be tried, one litter to each type, thus allowing 5 replications with 12 degrees of freedom for error. The main treatments (housings) would not be compared as accurately as the sub-treatments (sources of vitamin) for two reasons; less replication is provided, and litter differences are included in the error for evaluating the housing effects. Nevertheless, some information about housing may be got at little extra expense, and any interaction between housing and vitamin will be accurately evaluated.

In experiments on varieties or fertilizers on small plots, cultural practices with large machines may be tried on whole groups of the smaller plots, each group containing all the varieties. (Irrigation is one practice that demands large areas per treatment.) The series of cultural practices is usually replicated only a small number of times but the varieties are repeated on every cultural plot. Experiments of this type are called *split-plot*, the cultural *main plot* being split into smaller varietal *sub-plots*.

This design is also common in industrial research. Comparisons among relatively large machines, or comparisons of different conditions of temperature and humidity under which machines work, are *main plot* treatments, while adjustments internal to the machines are *sub-plot* treatments. Since the word *plot* is inappropriate in such applications, the designs are often called *nested*, in the sense of section 10.16.

The essential feature of the split-plot experiment is that the sub-plot treatments are not randomized over the whole large block but only over the main plots. Randomization of the sub-treatments is newly done in each main plot and the main treatments are randomized in the large blocks.

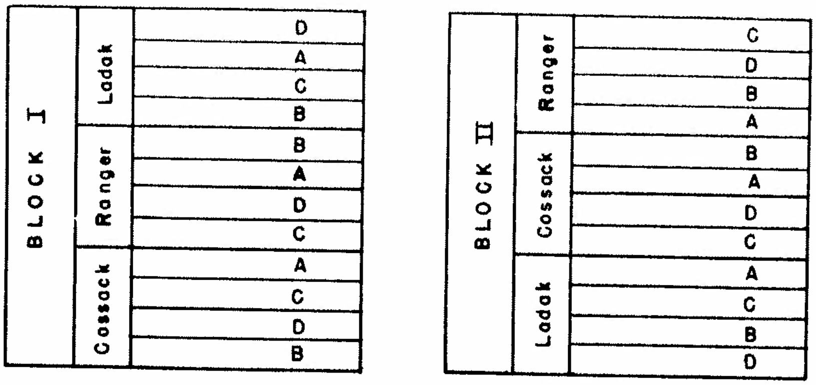

FIG. 12.12.1—First 2 blocks of split-plot experiment on alfalfa, illustrating random arrangement of main and sub-plots.

A consequence is that the experimental error for sub-treatments is different (characteristically smaller) than that for main treatments.

Figure 12.12.1 shows the field layout of a split-plot design with three varieties of alfalfa, the sub-treatments being four dates of final cutting (13). The first two harvests were common to all plots, the second on July 27, 1943. The third harvests were: *A*, none; *B*, September 1; *C*, September 20; *D*, October 7. Yields in 1944 are recorded in table 12.12.1. Such an experiment is, of course, not evaluated by a single season's yields; statistical methods for perennial crops are discussed in section 12.14.

In the analysis of variance the main plot analysis is that of randomized blocks with three varieties replicated in six blocks. The sub-plot analysis contains the sums of squares for dates of cutting, for the date × variety interactions, and for the sub-plot error, found by subtraction as shown at the foot of table 12.12.2.

The significant differences among dates of cutting were not unexpected, nor were the smaller yields following *B* and *C*. The last harvest should be either early enough to allow renewed growth and restoration of the consequent depletion of root reserves, or so late that no growth and depletion will ensue. The surprising features of the experiment were two; the yield following *C* being greater than *B*, since late September is usually considered a poor time to cut alfalfa in Iowa; and the absence of interaction between date and variety—Ladak is slow to renew growth after cutting and might have reacted differently from the other varieties.

In order to justify this analysis we need to study the model. In randomized blocks, the model for the split-plot or nested experiment is

$$X_{ijk} = \mu + M_i + B_j + \varepsilon_{ij} + T_k + (MT)_{ik} + \delta_{ijk}$$
$$i = 1 \ldots m,\ j = 1 \ldots b,\ k = 1 \ldots t,\ \varepsilon_{ij} = \mathcal{N}(0, \sigma_M),\ \delta_{ijk} = \mathcal{N}(0, \sigma_I)$$

Here, *M* stands for main plot treatments, *B* for blocks, and *T* for sub-plot treatments.

YIELDS OF THREE VARIETIE
FOUR D

Variety	Date
Ladak	A
	B
	C
	D
Cossack	A
	B
	C
	D
Ranger	A
	B
	C
	D
Total	

Variety	A
Ladak	11.25
Cossack	10.59
Ranger	10.22
Total	32.06
Mean (tons per acre)	1.78

The symbols *i*, *j* identif
within the main plot. The tv
to make the model realistic
consistently higher than thos
From the model, the error
plot treatments, say M_1 and

$\bar{\varepsilon}_{1\cdot}$

The $\bar{\varepsilon}$'s are averages over *b*
the variance of the mean di

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TABLE 12.12.1
YIELDS OF THREE VARIETIES OF ALFALFA (TONS PER ACRE) IN 1944 FOLLOWING FOUR DATES OF FINAL CUTTING IN 1943

Variety	Date	Blocks 1	2	3	4	5	6
Ladak	A	2.17	1.88	1.62	2.34	1.58	1.66
	B	1.58	1.26	1.22	1.59	1.25	0.94
	C	2.29	1.60	1.67	1.91	1.39	1.12
	D	2.23	2.01	1.82	2.10	1.66	1.10
		8.27	6.75	6.33	7.94	5.88	4.82
Cossack	A	2.33	2.01	1.70	1.78	1.42	1.35
	B	1.38	1.30	1.85	1.09	1.13	1.06
	C	1.86	1.70	1.81	1.54	1.67	0.88
	D	2.27	1.81	2.01	1.40	1.31	1.06
		7.84	6.82	7.37	5.81	5.53	4.35
Ranger	A	1.75	1.95	2.13	1.78	1.31	1.30
	B	1.52	1.47	1.80	1.37	1.01	1.31
	C	1.55	1.61	1.82	1.56	1.23	1.13
	D	1.56	1.72	1.99	1.55	1.51	1.33
		6.38	6.75	7.74	6.26	5.06	5.07
Total		22.49	20.32	21.44	20.01	16.47	14.24

Variety	Date of Cutting A	B	C	D	Total
Ladak	11.25	7.84	9.98	10.92	39.99
Cossack	10.59	7.81	9.46	9.86	37.72
Ranger	10.22	8.48	8.90	9.66	37.26
Total	32.06	24.13	28.34	30.44	114.97
Mean (tons per acre)	1.78	1.34	1.57	1.69	

The symbols i, j identify the main plot, while k identifies the sub-plot within the main plot. The two components of error, ε_{ij} and δ_{ijk}, are needed to make the model realistic: the sub-plots in one main plot often yield consistently higher than those in another, and ε_{ij} represents this difference. From the model, the error of the mean difference between two main plot treatments, say M_1 and M_2, is

$$\bar{\varepsilon}_{1\cdot} - \bar{\varepsilon}_{2\cdot} + \bar{\delta}_{1\cdot\cdot} - \bar{\delta}_{2\cdot\cdot}$$

The $\bar{\varepsilon}$'s are averages over b values, the $\bar{\delta}$'s over bt values. Consequently, the variance of the mean difference is between sub-plot treatments for a single main-plot treatment (e.g., between dates for Ladak).

In some experiments it is feasible to use either the split-plot design or ordinary randomized blocks in which the mt treatment combinations are randomized within each block. On the average, the two arrangements have the same overall accuracy. Relative to randomized blocks, the split-plot design gives reduced accuracy on the main-plot treatments and increased accuracy on sub-plot treatments and interactions. In some industrial experiments conducted as split-plots, the investigator apparently did not realize the implications of the split-plot arrangement and analyzed the design as if it were in randomized blocks. The consequences were to assign too low errors to main-plot treatments and too high errors to sub-plot treatments.

TABLE 12.12.3
PRESENTATION OF TREATMENT MEANS (TONS PER ACRE) AND STANDARD ERRORS

Variety	Date of Cutting ($\pm\sqrt{E_b/b} = \pm 0.0683$) A	B	C	D	Means
Ladak	1.875	1.307	1.664	1.820	1.667 ($\pm\sqrt{E_a/tb} = \pm 0.0753$)
Cossack	1.765	1.302	1.577	1.644	1.572
Ranger	1.704	1.414	1.484	1.610	1.553
Means	1.781	1.341	1.575	1.691	
		($\pm\sqrt{E_b/mb} = \pm 0.0394$)			

Care is required in the use of the correct standard errors for comparisons among treatment means. Table 12.12.3 shows the treatment means and *s.e.*'s for the alfalfa experiment, where $E_a = 0.1362$ and $E_b = 0.0280$ denote the main- and sub-plot Error mean squares. The *s.e.* ± 0.0683, which is derived from E_b, is the basis for computing the *s.e.* for comparisons that are part of the Variety-Date interactions and for comparisons among dates for a single variety or a group of the varieties. The *s.e.* ± 0.0753 for varietal means is derived from E_a. Some comparisons, for example those among varieties for Date A, require a standard error that involves both E_a and E_b, as described in (8).

Formally, the sub-plot error *S.S.* (45 *d.f.*) is the combined *S.S.* for the BT interactions (15 *d.f.*) and the BMT interactions (30 *d.f.*). Often, it is more realistic to regard Blocks as a random component rather than as a fixed component. In this case, the error for testing T is the BT mean square, while that for testing MT is the BMT mean square, if the two mean squares appear to differ.

Experimenters sometimes split the sub-plots and even the sub-sub-plots. The statistical methods are a natural extension of those given here. If T_1, T_2, T_3 denote the sets of treatments at three levels, the set T_1 are tested against the main-plot Error mean square, T_2 and the T_1T_2 interac-

TABLE 12.12.2
ANALYSIS OF VARIANCE OF SPLIT-PLOT EXPERIMENT ON ALFALFA

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square
Main plots:			
Varieties	2	0.1781 C	0.0890
Blocks	5	4.1499 B	0.8300
Main plot error	10	1.3622 D	0.1362
Sub-plots:			
Dates of cutting	3	1.9625 E	0.6542**
Date × variety	6	0.2105 F	0.0351
Sub-plot error	45	1.2586 see below	0.0280

GLM Printout

1. Correction: $C = (114.97)^2/72 = 183.5847$
2. Total: $(2.17)^2 + \ldots + (1.33)^2 - C = 9.1218$ A
3. Main plots: $\dfrac{(8.27)^2 + \ldots + (5.07)^2}{4} - C = 5.6902$ C + D + B
4. Varieties: $\dfrac{(39.99)^2 + \ldots + (37.26)^2}{24} - C = 0.1781$ C
5. Blocks: $\dfrac{(22.49)^2 + \ldots + (14.24)^2}{12} - C = 4.1499$ B
6. Main plot error: $5.6902 - (0.1781 + 4.1499) = 1.3622$ D
7. Sub-classes in variety-date table: $\dfrac{(11.25)^2 + \ldots + (9.66)^2}{6} - C = 2.3511$ C + E + F
8. Dates: $\dfrac{(32.06)^2 + \ldots + (30.44)^2}{18} - C = 1.9625$ E
9. Date × variety: $2.3511 - (0.1781 + 1.9625) = 0.2105$ F
10. Sub-plot error: $9.1218 - (5.6902 + 1.9625 + 0.2105) = 1.2586$
A − ((C + D + B) + E + F)

$$2\left(\frac{\sigma_M{}^2}{b} + \frac{\sigma_I{}^2}{bt}\right) = \frac{2}{bt}(\sigma_I{}^2 + t\sigma_M{}^2)$$

In the analysis of variance, the main plot Error mean square estimat $(\sigma_I{}^2 + t\sigma_M{}^2)$.

Consider now the difference $X_{ij1} - X_{ij2}$ between two sub-plots th are in the same main plot. According to the model,

$$X_{ij1} - X_{ij2} = T_1 - T_2 + (MT)_{i1} - (MT)_{i2} + \delta_{ij1} - \delta_{ij2}$$

The GLM Procedure

Class Level Information

Class	Levels	Values
variety	3	Cossack Ladak Ranger
date	4	A B C D
block	6	1 2 3 4 5 6

Number of observations 72

The GLM Procedure

Dependent Variable: yield

Source	DF	Sum of Squares		Mean Square	F Value	Pr > F
Model	26	7.86321944		0.30243152	10.81	<.0001
Error	45	1.25854583		0.02796769		
Corrected Total	71	9.12176528	A			

R-Square	Coeff Var	Root MSE	yield Mean
0.862028	10.47312	0.167235	1.596806

Source	DF	Type I SS		Mean Square	F Value	Pr > F
variety	2	0.17801944	C	0.08900972	3.18	0.0510
block	5	4.14982361	B	0.82996472	29.68	<.0001
variety*block	10	1.36234722	D	0.13623472	4.87	<.0001
date	3	1.96247083	E	0.65415694	23.39	<.0001
variety*date	6	0.21055833	F	0.03509306	1.25	0.2973

Source	DF	Type III SS	Mean Square	F Value	Pr > F
variety	2	0.17801944	0.08900972	3.18	0.0510
block	5	4.14982361	0.82996472	29.68	<.0001
variety*block	10	1.36234722	0.13623472	4.87	<.0001
date	3	1.96247083	0.65415694	23.39	<.0001
variety*date	6	0.21055833	0.03509306	1.25	0.2973