I. Comparing Group Means with Contrasts

Once we reject $H_0$ in the one-way ANOVA, we need to know which means are different. There are several additional types of hypotheses we may want to test:

1. $H_0 : \mu_i = \mu_i$.

2. $H_0 : \sum_{i=1}^{t} k_i \mu_i = 0$ where $\sum_{i=1}^{t} k_i = 0$

3. $H_0 : \mu_i = c$ (usually $c = 0$)
Ex. Midwest Heart Study

\[ \mu_1 = \text{mean cholesterol among current smokers} \]
\[ \mu_2 = \text{mean cholesterol among ex-smokers} \]
\[ \mu_3 = \text{mean cholesterol among never smokers} \]

What is the two-sided null hypothesis for testing

- current smokers versus ex-smokers?

- ever smokers versus never smokers?
Ex. Food safety – meat storage

“Shelf life” is the length of time a pre-packaged cut of meat remains safe, nutritious, and salable. Standard packing (styrofoam tray with plastic wrap) has a shelf life of about 48 hours, after which meat quality begins to deteriorate from bacterial growth, discoloration, and shrinkage.

Four packaging methods were studied:

$\mu_1 =$ standard packaging, mean bacterial growth

$\mu_2 =$ vacuum packaging, mean bacterial growth

$\mu_3 =$ packaging with a gas mixture, mean bacterial growth

$\mu_4 =$ packaging with a pure CO$_2$ gas, mean bacterial growth
What is the two-sided null hypothesis for testing:

- added gas versus no added gas?

- new packaging versus standard packing?

- pure CO$_2$ versus gas mixture?
(1) Comparing two means to each other

Testing $H_0 : \mu_i = \mu_{i'}$ vs. $H_A : \mu_i \neq \mu_{i'}$ (or any one-sided test of two means) is just like a t-test. Since the one-way ANOVA assumes

$$Y_{ij} \overset{\text{iid}}{\sim} N(\mu_i, \sigma^2)$$

then

$$\bar{Y}_{i} \overset{\text{iid}}{\sim} \cdot$$

and thus

$$\bar{Y}_{i} - \bar{Y}_{i'} \overset{\text{iid}}{\sim} \cdot$$
For a two-sided test, we reject $H_0 : \mu_i = \mu_{i'}$ versus $H_A : \mu_i \neq \mu_{i'}$ at level $\alpha$ if

$$t^* = \frac{|\bar{Y}_i - \bar{Y}_{i'}|}{\sqrt{\text{MSE} \left( \frac{1}{r_i} + \frac{1}{r_{i'}} \right)}} > t_{\alpha/2, df}$$

or if

$$\bar{Y}_i - \bar{Y}_{i'} \pm t_{\alpha/2, df} \sqrt{\text{MSE} \left( \frac{1}{r_i} + \frac{1}{r_{i'}} \right)}$$

does not cover 0. What $df$ do we use?

One-sided tests and intervals are constructed similarly.
(2) Comparing a contrast to a constant

In the notation of the book:

\[ C = \sum_{i=1}^{t} k_i \mu_i \]

true contrast

\[ c = \hat{C} = \sum_{i=1}^{t} k_i \hat{\mu}_i = \sum_{i=1}^{t} k_i \overline{Y}_i. \]

estimated contrast

In order to be a contrast, we need \( \sum_{i=1}^{t} k_i = \)

We also need to know

\[ \text{V} \hat{\text{a}} \text{r} [\hat{C}] = \text{V} \hat{\text{a}} \text{r} \left[ \sum_{i} k_i \hat{\mu}_i \right] = \]
Because

$$\hat{C} = \sum_{i=1}^{t} k_i \bar{Y}_i.$$ 

$$\text{Vâr}[\hat{C}] = \text{MSE} \sum_{i=1}^{t} \frac{k_i^2}{r_i}$$

we can expand our pairwise comparison in (1) above to get

$$\hat{C} = \sum_{i=1}^{t} k_i \bar{Y}_i. \sim$$
Thus we reject $H_0 : C = 0$ vs $H_A : C \neq 0$ at level $\alpha$ if

$$t^* = \frac{|\hat{C}|}{\sqrt{\text{Vâr}[\hat{C}]}} > t_{\alpha/2, df}$$

or if $\hat{C} \pm t_{\alpha/2, df} \sqrt{\text{Vâr}[\hat{C}]}$ does not cover 0. What $df$ do we use?

One-sided tests and confidence intervals are constructed similarly.
We can also compute how much any particular contrast contributed to the rejection of $H_0$ (equivalently, to the size of SST). This is the contrast sum of squares and is defined as:

\[
SSC = \frac{\hat{C}^2}{\sum_{i=1}^{t} k_i^2 r_i} = \left( \frac{\sum_{i=1}^{t} k_i \bar{Y}_i}{\sum_{i=1}^{t} k_i^2 r_i} \right)^2
\]

and has $df=1$. 
The SSC values for a complete set of orthogonal contrasts have a special property.

What are orthogonal contrasts? Consider two contrasts

\[ \hat{C} = \sum_{i=1}^{t} k_i \bar{Y}_i \] and \[ \hat{D} = \sum_{i=1}^{t} d_i \bar{Y}_i \]. When are they orthogonal?

What is a complete set of orthogonal contrasts?

What is the special property? (We need balance: all \( r_i \) equal.)
**Ex.** Midwest Heart Study

Write down a complete set of orthogonal contrasts. Assume

\[ r_1 = r_2 = r_3. \]

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**Ex.** Food storage study

Write down a complete set of orthogonal contrasts. Assume

\[ r_1 = r_2 = r_3 = r_4. \]
With a complete set of orthogonal contrasts, we can write down an ANOVA table that partitions the SST into $t - 1$ independent SSC:

<table>
<thead>
<tr>
<th>Source</th>
<th>df</th>
<th>SS</th>
<th>MS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Contrast 1</td>
<td>1</td>
<td>$(\hat{C}_1)^2 / \sum k_i^2 / r$</td>
<td>$SSC_1$</td>
</tr>
<tr>
<td>Contrast 2</td>
<td>1</td>
<td>$(\hat{C}_2)^2 / \sum k_i^2 / r$</td>
<td>$SSC_2$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Contrast $t - 1$</td>
<td>1</td>
<td>$(\hat{C}_{t-1})^2 / \sum k_i^2 / r$</td>
<td>$SSC_{t-1}$</td>
</tr>
<tr>
<td>Error</td>
<td>$N - t$</td>
<td>$\sum_i \sum_j (Y_{ij} - \bar{Y}_i)^2$</td>
<td>$SSE / (N - t)$</td>
</tr>
<tr>
<td>Total</td>
<td>$N - 1$</td>
<td>$\sum_i \sum_j (Y_{ij} - \bar{Y}_c)^2$</td>
<td></td>
</tr>
</tbody>
</table>
Can we construct an F-test using the MSC?

To test $H_0 : C = 0$ vs $H_A : C \neq 0$ reject at level $\alpha$ if

$$F^* = \frac{MSC}{MSE} > F_{\alpha,1,N-t}$$

Does $F^*$ here equal $(t^*)^2$ from above?

Can we do a one-sided F-test?
Why does having an orthogonal set of contrasts help us when we want to test multiple hypotheses?

However, the contrasts in an orthogonal set do not necessarily test scientifically sensible or meaningful hypotheses. Contrasts should be primarily constructed to answer the questions of interest.

Coding: SAS – CONTRAST or ESTIMATE statement in PROC GLM
S-PLUS & R – contrasts option in lm() function
One special kind of a contrast is a polynomial contrast, which can capture, e.g., any quadratic trend in the response across the group means.

**Ex.** Grain production & plant density (text p.84)

A plot of land is divided into 15 subplots. 5 treatments are each randomly assigned to 3 subplots. The “treatment” is the density at which seeds are planted in each subplot: 10, 20, 30, 40, or 50. After the growing season, the grain is harvested from each subplot and measured.

Is there a trend in the grain yield as the densities increase?
Option #1: Fit the data as a regression of grain yield on density with polynomial terms for density.

Option #2: Fit the data as a one-way ANOVA of grain yield on density and construct polynomial contrasts for density.

Do we get the same:

- sums of squares?
- test statistics for the overall F-test?
- test statistics for the linear test, quadratic test, etc.?
In summary,

- Polynomial contrasts only make sense when the factor levels are ordered and they correspond to specific quantitative levels. For example, levels of low, medium, and high is not sufficient.
- ANOVA with polynomial contrasts is exactly the same as a polynomial regression when the group sample sizes are equal.
- If the group sample sizes are unequal, use a polynomial regression and not an ANOVA.
- If your factor levels are ordered but qualitative (e.g., low, medium, high), resist the temptation to assign quantitative numbers such as low=1, medium=2, high=3. Your results will depend completely on which numbers you happened to use!!
(3) Comparing one mean to a constant

Testing $H_0 : \mu_i = c$ vs. $H_A : \mu_i \neq c$ for any constant $c$, we reject $H_0$ at level $\alpha$ if

$$t^* = \frac{|\bar{Y}_i - c|}{\sqrt{\text{MSE} / r_i}} > t_{\alpha/2, df}$$

or if $\bar{Y}_i \pm t_{\alpha/2, df} \sqrt{\text{MSE} / r_i}$ does not cover 0. What $df$ do we use?

One-sided tests and confidence intervals are constructed similarly.