Martingale

stochastic process without drift.

Simplest (and earliest) def'n:

\( \mathcal{F}_n \) \( n=0,1,\ldots \) \( \exists \) is a martingale if

i) \( \mathbb{E}[X_n] < \infty \ \forall n \)

ii) \( \mathbb{E}[X_{n+1} | X_0, X_1, \ldots, X_n] = X_n \ \forall n \)

More complex def'n.

\( \exists Y_n \) is a (possibly multivariate) stoch. process.

Let \( \mathcal{F}_n = \sigma \{ Y_0, Y_1, \ldots, Y_n \} \)

\( \exists Y_n \) is a mart. wrt \( \{ Y_n \} \) equiv. wrt \( \{ \mathcal{F}_n \} \) if.

i) \( \mathbb{E}[X_n] < \infty \ \forall n \)

ii) \( X_n \) is deterministic fn. of \( Y_0, Y_1, \ldots, Y_n \); \( X_n \in \mathcal{F}_n \)

iii) \( \mathbb{E}[X_{n+1} | Y_0, Y_1, \ldots, Y_n] = X_n \)

Think of \( Y_0, Y_1, \ldots, Y_n \) (i.e. \( \mathcal{F}_n \)) as the information available, history up through stage \( n \).

History may include covariates, as well as \( X \)'s themselves.
Review of Conditional Expectation

The following list summarizes these and other properties of conditional expectations. Here, with or without affixes, $X$ and $Y$ are random variables, $c$ is a real number, $g$ is a function for which $E[|g(X)|] < \infty$, $f$ is a bounded function and $h$ is a function of two variables for which $E[|h(X, Y)|] < \infty$.

$$E[a_1 g(X_1) + a_2 g(X_2)|Y] = a_1 E[g(X_1)|Y] + a_2 E[g(X_2)|Y],$$  \hspace{1cm} (1.6)

$$g \geq 0 \implies E[g(X)|Y] \geq 0,$$  \hspace{1cm} (1.7)

$$E[h(X, Y)|Y = y] = E[h(X, y)|Y = y],$$  \hspace{1cm} (1.8)

$$E[g(X)|Y = y] = E[g(X)]$$  \hspace{1cm} (1.9)

if $X$ and $Y$ are independent, and

$$E[g(X)f(Y)|Y] = f(Y)E[g(X)|Y],$$  \hspace{1cm} (1.10)

and

$$E[g(X)f(Y)] = E[E[g(X)|Y]f(Y)].$$  \hspace{1cm} (1.11)

As consequences of (1.6), (1.10) and (1.11), with either $g \equiv 1$ or $f \equiv 1$, we obtain,

$$E[c|Y] = c,$$  \hspace{1cm} (1.12)

$$E[f(Y)|Y] = f(Y),$$  \hspace{1cm} (1.13)

and

$$E[g(X)] = E[E[g(X)|Y]].$$  \hspace{1cm} (1.14)

- **Jensen's conditional inequality** \textit{KoT p.249}

  Let $\phi$ be a convex function.

  $$E[\phi(X)|Y_0, \ldots, Y_n] \geq \phi(E[X|Y_0, \ldots, Y_n])$$

  corollary: $E[1E(X|Y_0, \ldots, Y_n)]^p \leq E[1^p X]$ for $p > 1$

- **Law of total prob. for conditional expectation.**

  Suppose $E|X| < \infty$

  $$E[X|Z] = E[E[X|Y, Z]|Z].$$  \textit{KoT, p.246}

Footnote.
Classic Example

Let $Y_1, Y_2, \ldots$ be indep. $E(Y_n) = 0$, $E|Y_n| < \infty \forall n$.

Define $Y_0 = 0$. Let $X_n = \sum_{i=0}^{n} Y_i$.

Then $\{X_n\}$ is a mart. wrt. $\mathcal{F}_n$.

Proof: verify properties

1) $E[X_n] = E[Y_0 + Y_1 + \ldots + Y_n]
\leq \sum_{i=0}^{n} E|Y_i| < \infty$ by assumption.

2) $E(X_{n+1} | Y_0, \ldots, Y_n) = E(X_n + Y_{n+1} | Y_0, \ldots, Y_n)
= E(X_n | Y_0, \ldots, Y_n) + E(Y_{n+1} | Y_0, \ldots, Y_n)
= X_n + E(Y_{n+1})$

a) because $X_n$ is fn of $Y_0, \ldots, Y_n$ + 1.13

b) because of indep. 1.9

$= X_n$ because $E(Y_{n+1}) = 0$. 

Properties

1) Generalization of martingale property
\[ E[X_{n+k} | Y_0, ..., Y_n] = (E[X_{n+k} | \mathcal{F}_n]) = X_n \quad \text{for } k \geq 1 \]


Martingales in general share 2 important properties w. sums of independent mean 0 rvs:

2) Constant mean - no drift
\[ E(X_n) = E(X_0) \forall n \]

3) Uncorrelated increments (but maybe dependent).
   Consider integers \( S < t \leq u < V \)
\[ \text{Cov}(X_v - X_u, X_t - X_S) = 0 \]

pf: \[ \text{Cov}(X_v - X_u, X_t - X_S) = E \left[ (X_v - X_u)(X_t - X_S) \right] \]

In proofs for marts., trick is to condition on past but as far forward as possible:
\[ E[(X_v - X_u)(X_t - X_S)] = E \{ E[(X_v - X_u)(X_t - X_S) | \mathcal{F}_u] \} \]

\[ = E E[(X_t - X_S)E(X_v - X_u | \mathcal{F}_u)] \]

but \( E(X_v - X_u | \mathcal{F}_u) = E(X_v | \mathcal{F}_u) - E(X_u | \mathcal{F}_u) = X_u - X_u = 0 \).
2) Doob's martingale \textcopyright{} p. 246

Accumulating data about a random variable.

Let $X$ be rv with $E|X| < \infty$.

Let $Y_1, Y_2, \ldots$ be stochastic process.

Define $X_n = E[X | Y_1, \ldots, Y_n]$.

Then $\sum_{n=1}^{\infty} X_n$ is a Doob's martingale w.r.t. $\mathcal{F}_n$.

Doob's martingale useful in Stochastic curtailment.
Optional Sampling/Stopping of Mart

Let $\xi X_n \xi$ be martingale wrt $\xi V_n \xi$.

Broadest sense: Let $T_1 \leq T_2 \leq \cdots \leq T_n \leq \cdots$ be increasing sequence of stopping times.

Theorems give regularity conditions under which

$X_{T_1}, X_{T_2}, \ldots$ (martingale sampled at random times)

is still a martingale.

Narrower focus: Single stopping time, $T$

Reg. conditions under which $E(X_T) = E(X_0)$

i.e. $\xi X_0, X_T \xi$ forms a 2 element martingale recall constant mean of martingale.

Regularity conditions involve some sort of boundedness conditions on $T$ and/or $\xi X_n \xi$. 
Doob's Optimal Stopping Theorem
Th 3.2 K+T (1st course, p. 261).

Let $\{X_n\}$ be a mart. $T$ a stopping time
If:

a) $P_n \{ T < \infty \} = 1$

b) $E |X_T| < \infty$

c) $\lim_{n \to \infty} E [X_n \mathbf{1}_{\{T > n\}}] = 0$

i.e. $\lim_{n \to \infty} \int_{T > n} X_n dP = 0$

Then $E[X_T] = E[X_0]$.  

Note:
1) K+T emphasize that (b) is not a consequence of $E |X_n| < \infty \forall n$; must be checked.

Doob (1953, Stochastic Processes, p. 302-303) proves that if $P_n \{ T < \infty \} = 1$
then $L.U.B$ $E[X_n |13 < \infty]$ is sufficient for $E |X_T| < \infty$. (L.U.B = least upper bound = supremum)

2) c) has variants

common $\lim_{n \to \infty} \inf_{s \geq n} E [|X_s| \mathbf{1}_{\{T > n\}}] = 0.$

Fristedt & Gray, 1997, p. 467
Simpler Optional Stopping Thms.

Restrictions on $T$ get less while those on $\{X_n\}$ get stricter as we go down the list.

1) $T$ is bounded a.s. $\exists M < \infty$ s.t. $P_n (T < M) = 1$

Frystyk & Gray, Prop. 9, p. 469.

2) $E(T) < \infty$ & $\exists K < \infty$ s.t.

$$E \left[ |X_{n+1} - X_n| \mid Y_0, y_1, \ldots, Y_n \right] \leq K$$

w. prob. 1

(i.e. for almost all $w$) in the set $[T > n]$

$k + T$, p. 260

Note: Here we assume that $\{X_n\}$ is a mart + $T$ is a stopping time w. respect to $\{Y_n\}$.

Williams, Probability with martingales, p. 120

has stronger condition $|X_{n+1} - X_n| \leq K$ w.

3) $P_n (T < \omega) = 1 > E (\sup_n |X_n|) < \infty$

Ash, 1972, p. 305.

$P_n (T < \omega) = 1 > E (\sup_n |X_{\text{min}(T, n)}|) < \infty$

$k + T$, Th. 3.1, p. 259

Williams (1991), p. 106 has stronger condition $|X_n| \leq K$ w.

Under any of these conditions

$E(X_T) = E(X_0)$
Optional Stopping Counter-Example.

Martingale + stopping time with \( E(X_T) \neq E(X_0) \).

Let \( Y_0 = 0 \), \( Y_1, Y_2, \ldots \), iid

\[
\Pr \{ Y_k = +1 \} = p \quad \Pr \{ Y_k = -1 \} = \frac{1}{2}
\]

Set \( X_n = \sum_{i=0}^{n} Y_n \)

\( \{ X_n \} \) is a m.a.r.t. wrt \( \{ Y_n \} \)

symmetric random walk.

Let \( T = \inf_{n \geq 1} \{ n : X_n = 13 \} \) a stopping time.

Note: \( E(X_n) = E(X_0) = 0 \) \( \forall n \)

but \( X_T = 1 \)

So \( E(X_T) = 1 \)

\( E(X_T) \neq E(X_0) \)
Properties of counter-example.

1) \( P(T < \infty) = 1 \) finite termination.

This situation is equivalent to the gambler's ruin problem where an individual with stake = 1 plays against infinitely rich adversary.

Prob. of ultimately entering absorbing state of 0 (gambler's ruin) = 1.

(KT first course p. 93-94)

2) \( E|X_T| < \infty \)

So 1 & 2 mean conditions a) & b) of Doob's optional stopping theorem are met.

Whittle (2000, ed. 4, p. 304) points out but does not prove that condition c is violated:

"The process can show infinite excursions, (necessarily in a negative direction) before the stopping set \( X=1 \) is attained."