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Large Point-referenced Datasets

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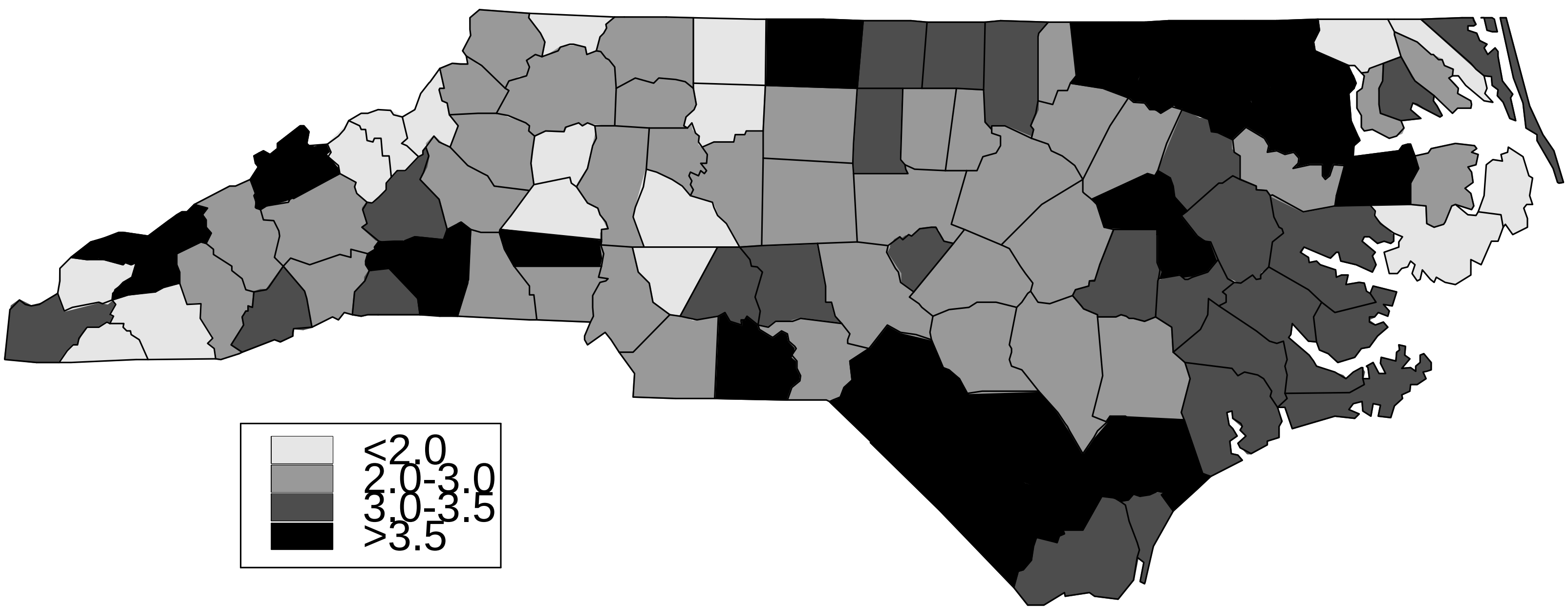
- Inference for new areal units?
- Descriptive/algorithmic vs. Model-based

Areal unit data

Actual Transformed SIDS Rates



Proximity matrices

- W, entries w_{ij} (with $w_{ii} = 0$). Choices for w_{ij}:
  - $w_{ij} = 1$ if i, j share a common boundary (possibly a common vertex)
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- Could also define first-order neighbors $W^{(1)}$, second-order neighbors $W^{(2)}$, etc.

Measures of spatial association

- Moran's I: essentially an "areal covariogram"

$$I = \frac{n \sum_i \sum_j w_{ij}(Y_i - \bar{Y})(Y_j - \bar{Y})}{(\sum_{i \neq j} w_{ij}) \sum_i (Y_i - \bar{Y})^2}$$

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  Moran has mean $-1/(n-1) \approx 0$, Geary has mean 1
- Significance testing by comparing to a collection of say 1000 random permutations of the Y_i

Measures of spatial association (cont'd)

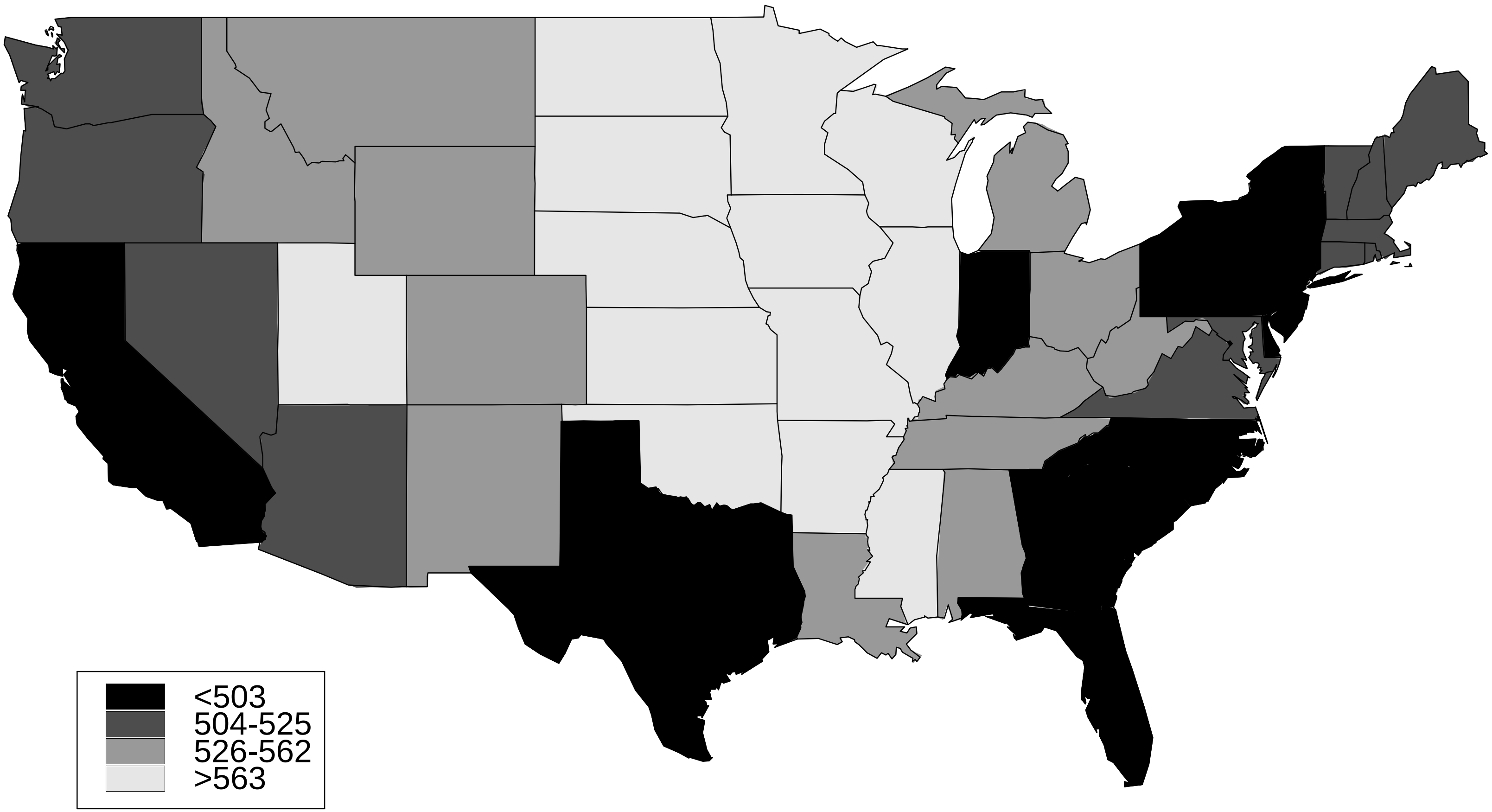

Figure 1: Choropleth map of 1999 average verbal SAT scores, lower 48 U.S. states.

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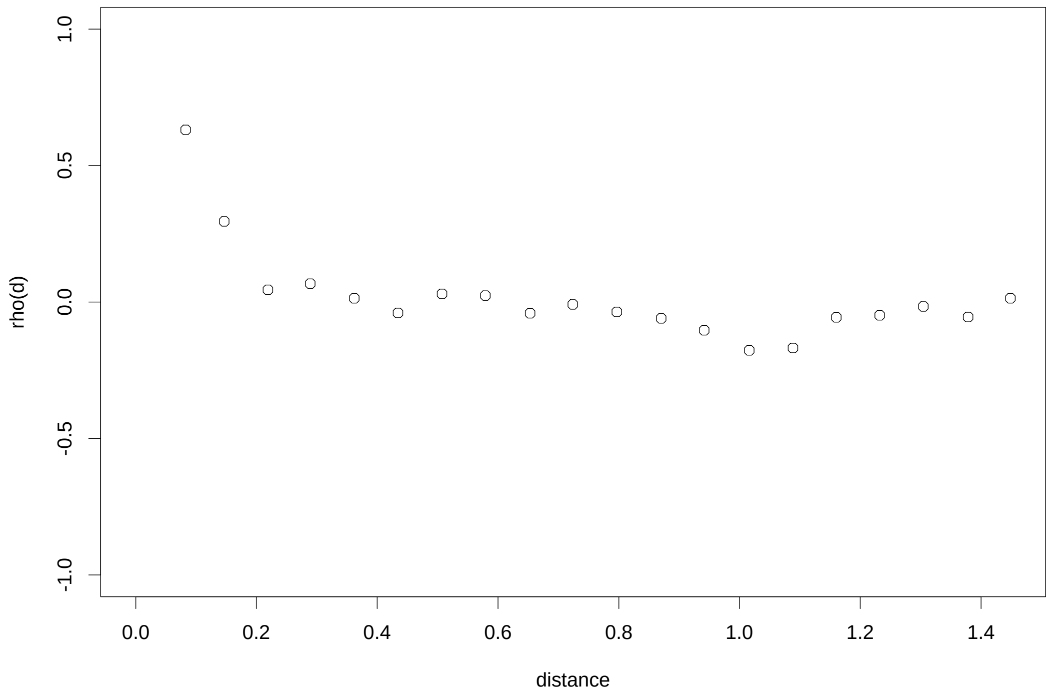
- Warning: These data have not been adjusted for covariates, such as the proportion of students who take the exam (Midwestern colleges have historically relied on the ACT, not the SAT; only the best and brightest students in these states would bother taking the SAT)
- $\Rightarrow$ the map, I, and C all motivate the search for spatial covariates!

Correlogram (via Moran's I)



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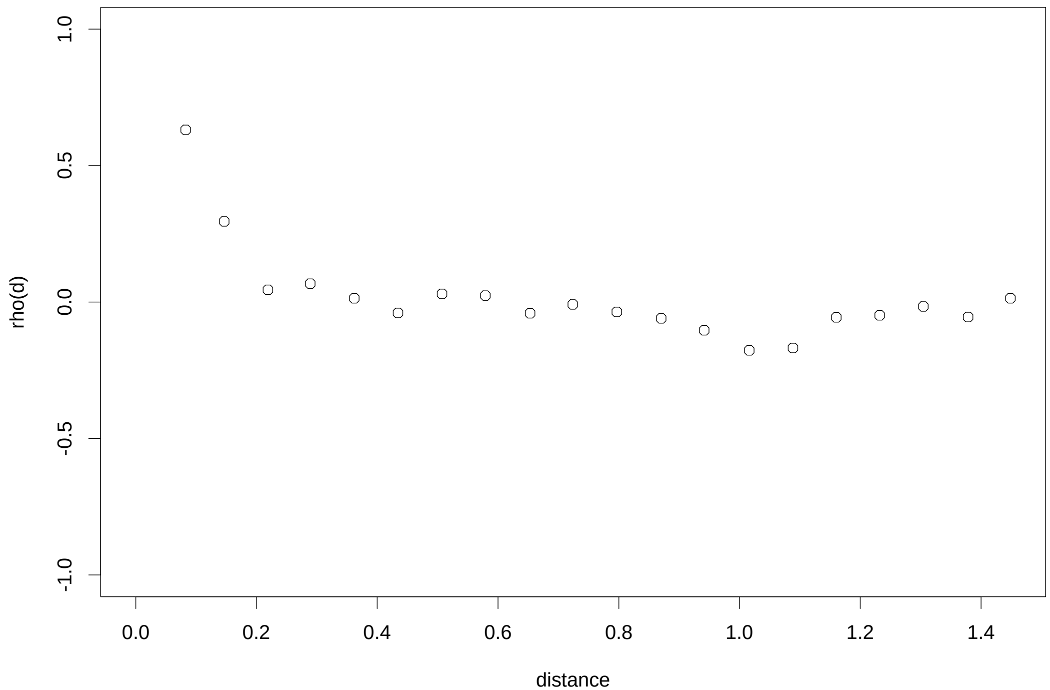

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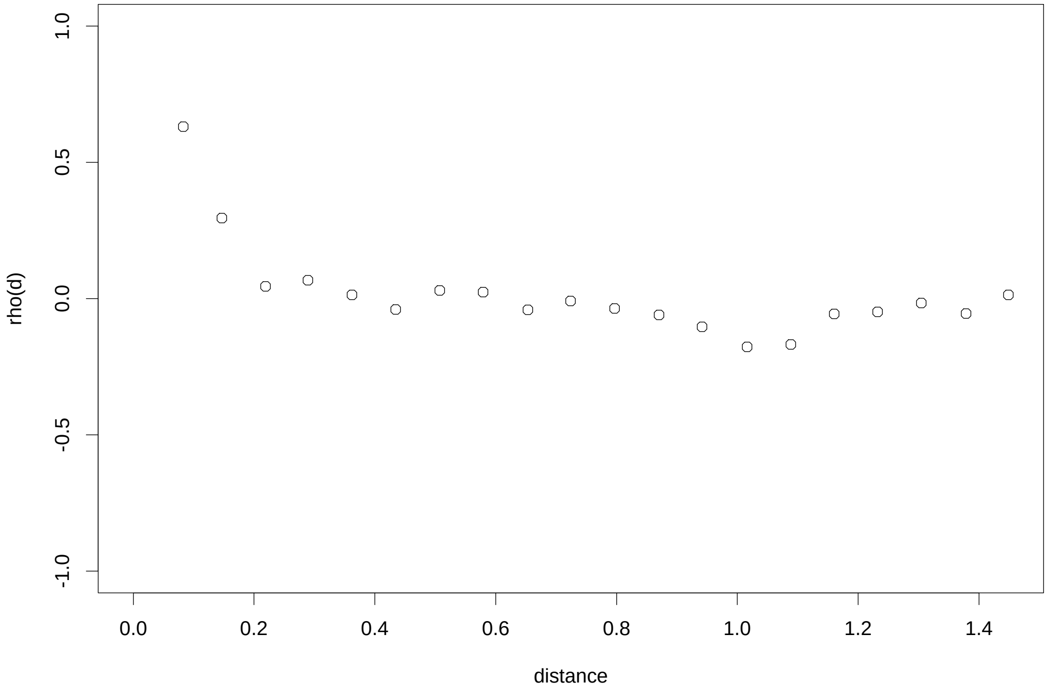

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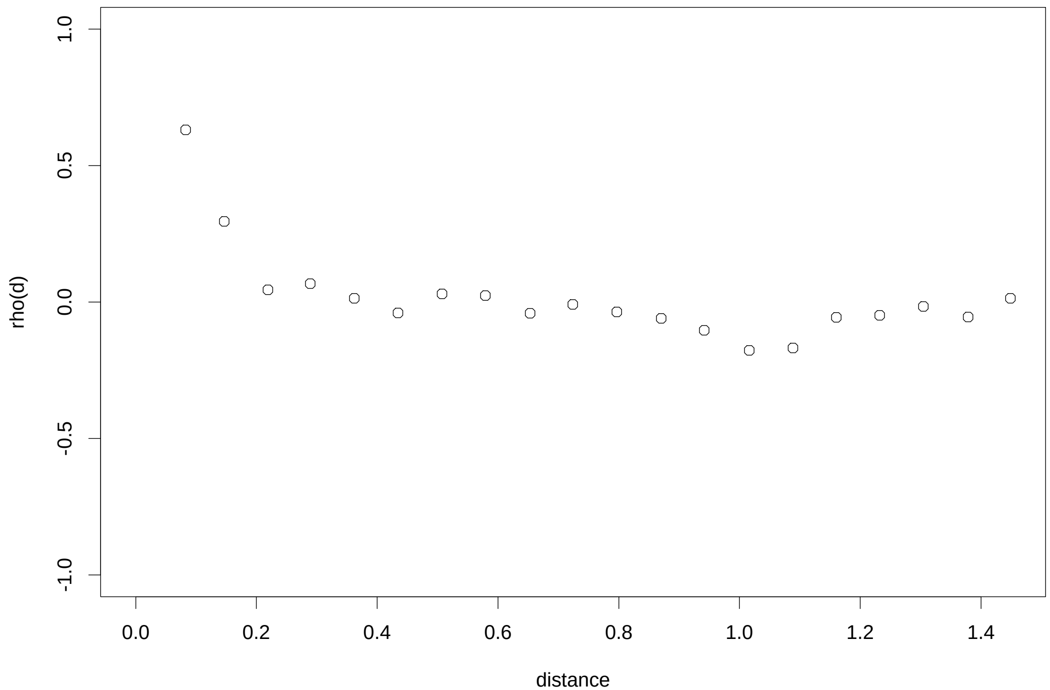

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- spatial analogue of the temporal lag autocorrelation plot

Spatial smoothers

- To smooth Y_i, replace with $\hat{Y}_i = \frac{\sum_i w_{ij} Y_j}{w_{i+}}$
- More generally, we could include the value actually observed for unit i, and revise our smoother to

$$(1-\alpha)Y_i + \alpha\hat{Y}_i$$

For $0 < \alpha < 1$, this is a linear (convex) combination in "shrinkage" form

Finally, we could try model-based smoothing, i.e., based on $E(Y_i|Data)$, i.e., the mean of the predictive distribution. Smoothers then emerge as byproducts of the hierarchical spatial models we use to explain the Y_i's

Conditional autoregressive (CAR) model

- Gaussian (autonormal) case

$$p(y_i|y_j, j \neq i) = N\left(\sum_j b_{ij} y_j, \tau_i^2\right)$$

Using Brook's Lemma we can obtain

$$p(y_1, y_2, ...y_n) \propto \exp\left\{-\frac{1}{2}\mathbf{y}' D^{-1}(I - B)\mathbf{y}\right\}$$

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- $D^{-1}(I - B)$ symmetric requires $\frac{b_{ij}}{\tau_i^2} = \frac{b_{ji}}{\tau_j^2}$ for all i, j

CAR Model (cont'd)

- Returning to W, let $b_{ij} = w_{ij}/w_{i+}$ and $\tau_i^2 = \tau^2/w_{i+}$, so

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- Not a data model, a random effects model!

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- $\rho = 0$ interpretable as independence

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- So, used with random effects, scope of spatial pattern may be limited

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- Non-Gaussian case: For binary data, the autologistic:

$$p(y_i|y_j, j \neq i) \propto \exp\left\{ \phi \sum_j w_{ij} I(y_i = y_j) \right\}$$