

Introduction to spatial data and models

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 - highly multivariate, with many important predictors and response variables,
 - geographically referenced, and often presented as maps, and
 - temporally correlated, as in longitudinal or other time series structures.
- ⇒ motivates *hierarchical* modeling and data analysis for complex spatial (and spatiotemporal) data sets.

Type of spatial data

- *point-referenced data*, where $Y(\mathbf{s})$ is a random vector at a location $\mathbf{s} \in \mathcal{R}^r$, where \mathbf{s} varies *continuously* over D , a fixed subset of \mathcal{R}^r that contains an r -dimensional rectangle of positive volume;

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- *areal data*, where D is again a fixed subset (of regular or irregular shape), but now partitioned into a **finite** number of areal units with well-defined boundaries;
- *point pattern data*, where now D is itself random; its index set gives the locations of random events that are the spatial point pattern. $Y(\mathbf{s})$ itself can simply equal 1 for all $\mathbf{s} \in D$ (indicating occurrence of the event), or possibly give some additional covariate information (producing a *marked point pattern process*).

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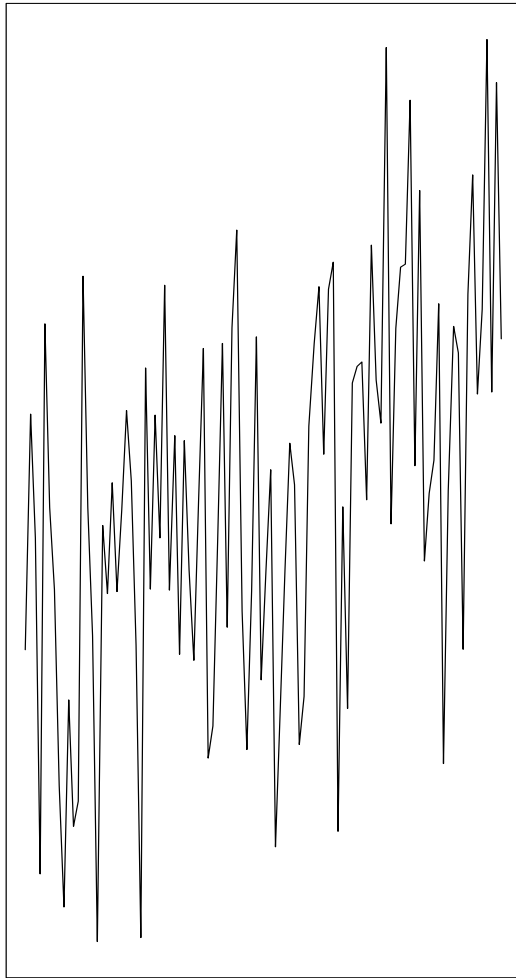
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- EDA tools examine both first and second order behavior

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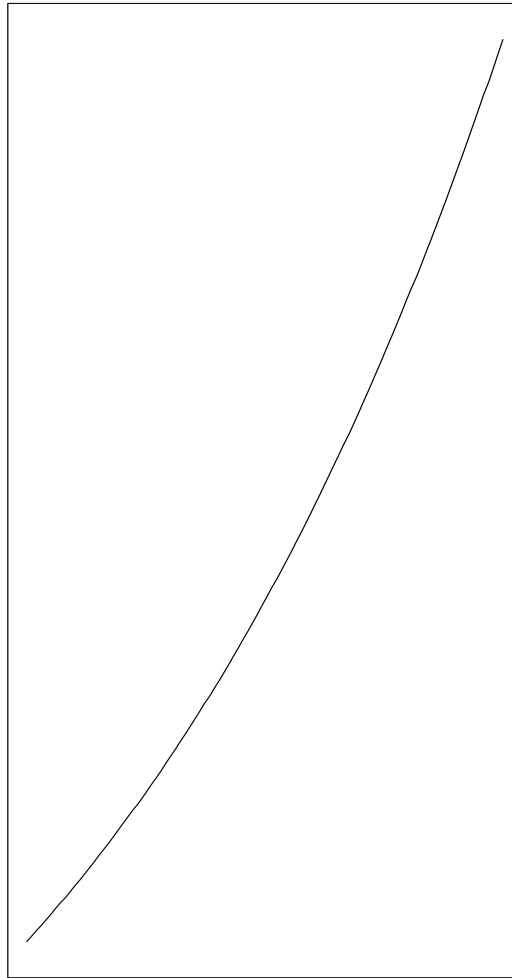
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- Preliminary displays: Simple locations to surface displays

First Law of Geography



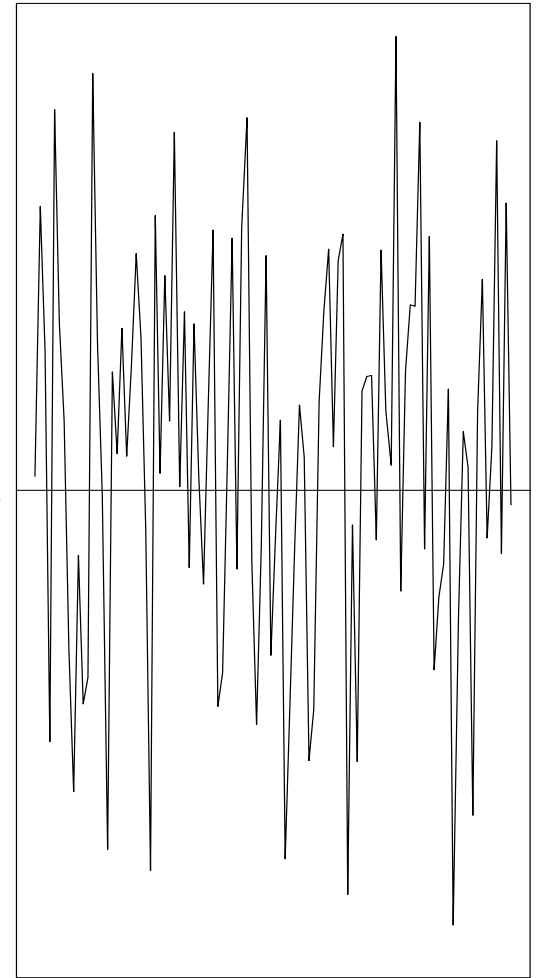
data

=



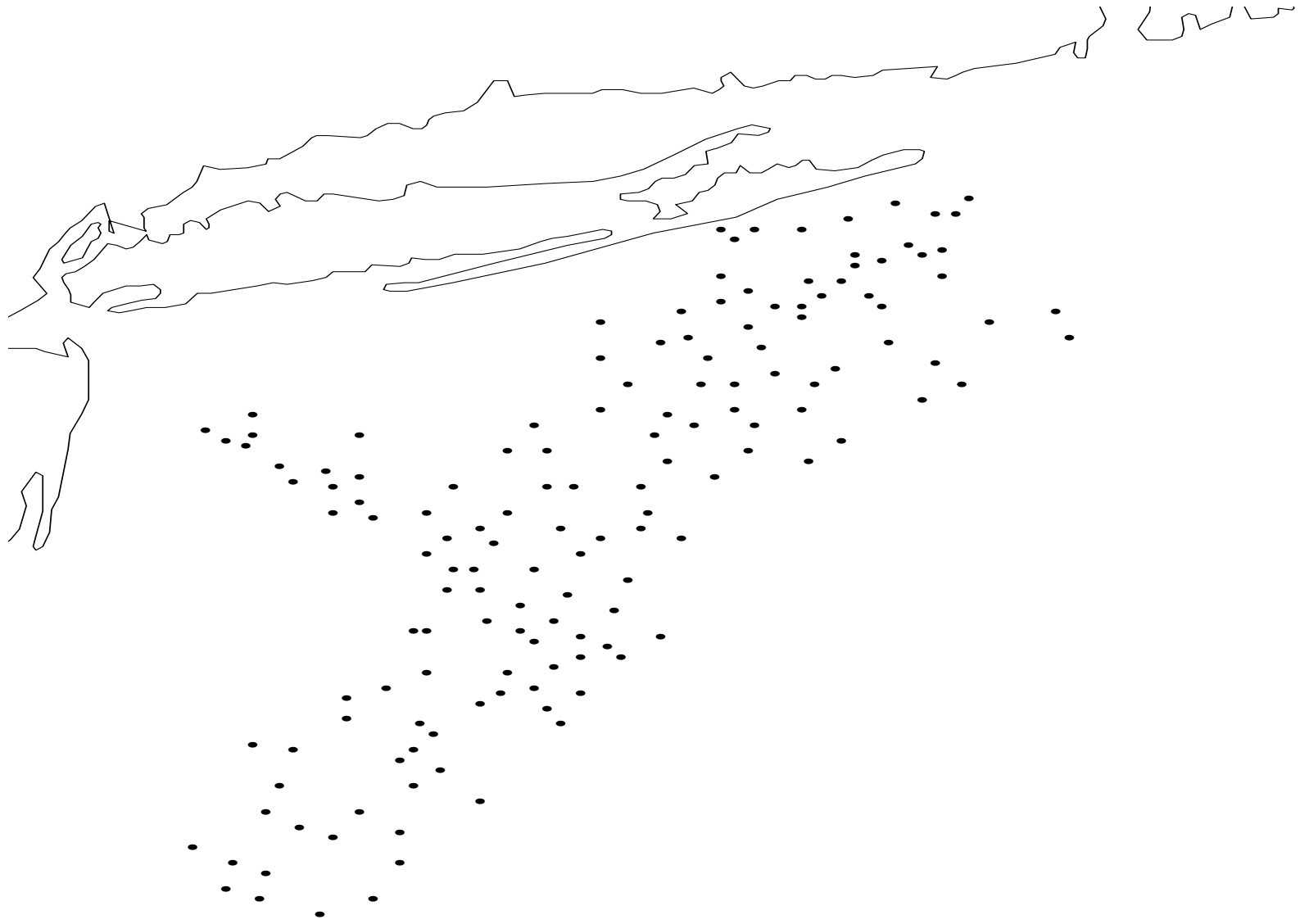
mean

+



error

Scallops Sites



Deterministic Surface Interpolation

- Spatial surface observed at finite set of locations
 $\mathcal{S} = \{s_1, s_2, \dots, s_n\}$

- Tessellate the spatial domain (usually with data locations as vertices)

- Fit an interpolating polynomial:

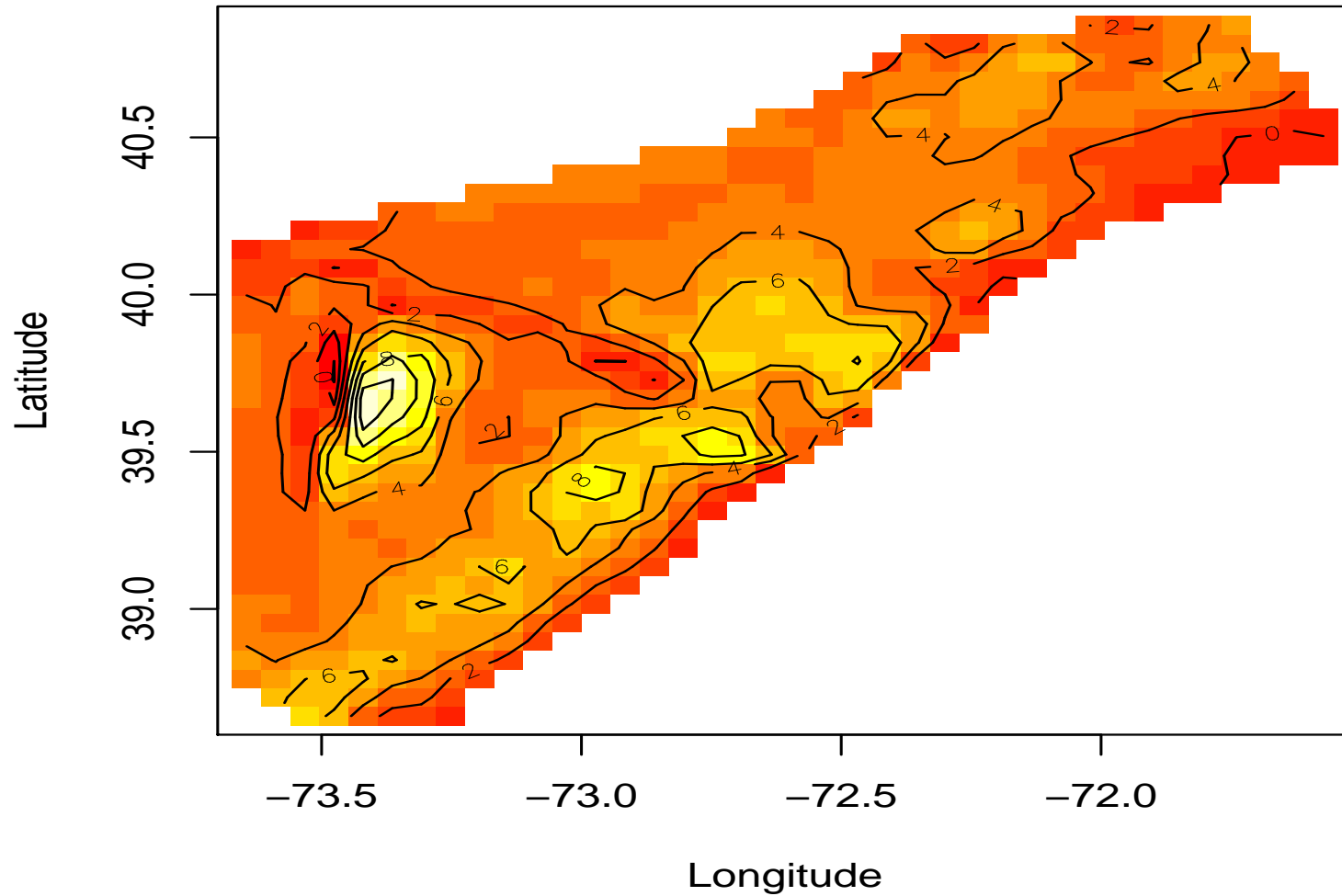
$$f(\mathbf{s}) = \sum_i w_i(\mathcal{S}; \mathbf{s}) f(\mathbf{s}_i)$$

- “Interpolate” by reading off $f(\mathbf{s}_0)$.

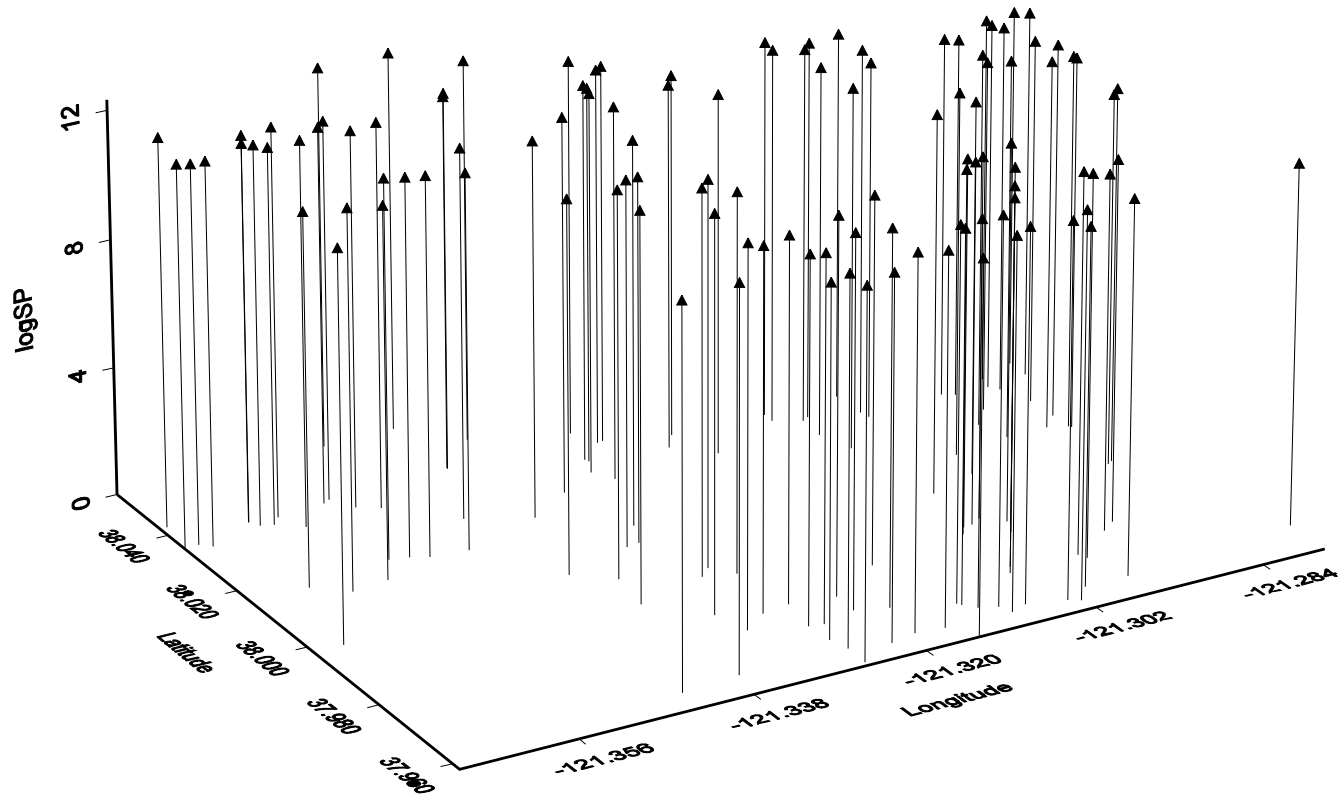
- Issues:

- Sensitivity to tessellations
- Choices of multivariate interpolators
- Numerical error analysis

Scallops Data: Image and contour plot



Drop-line scatter plot



Surface plot

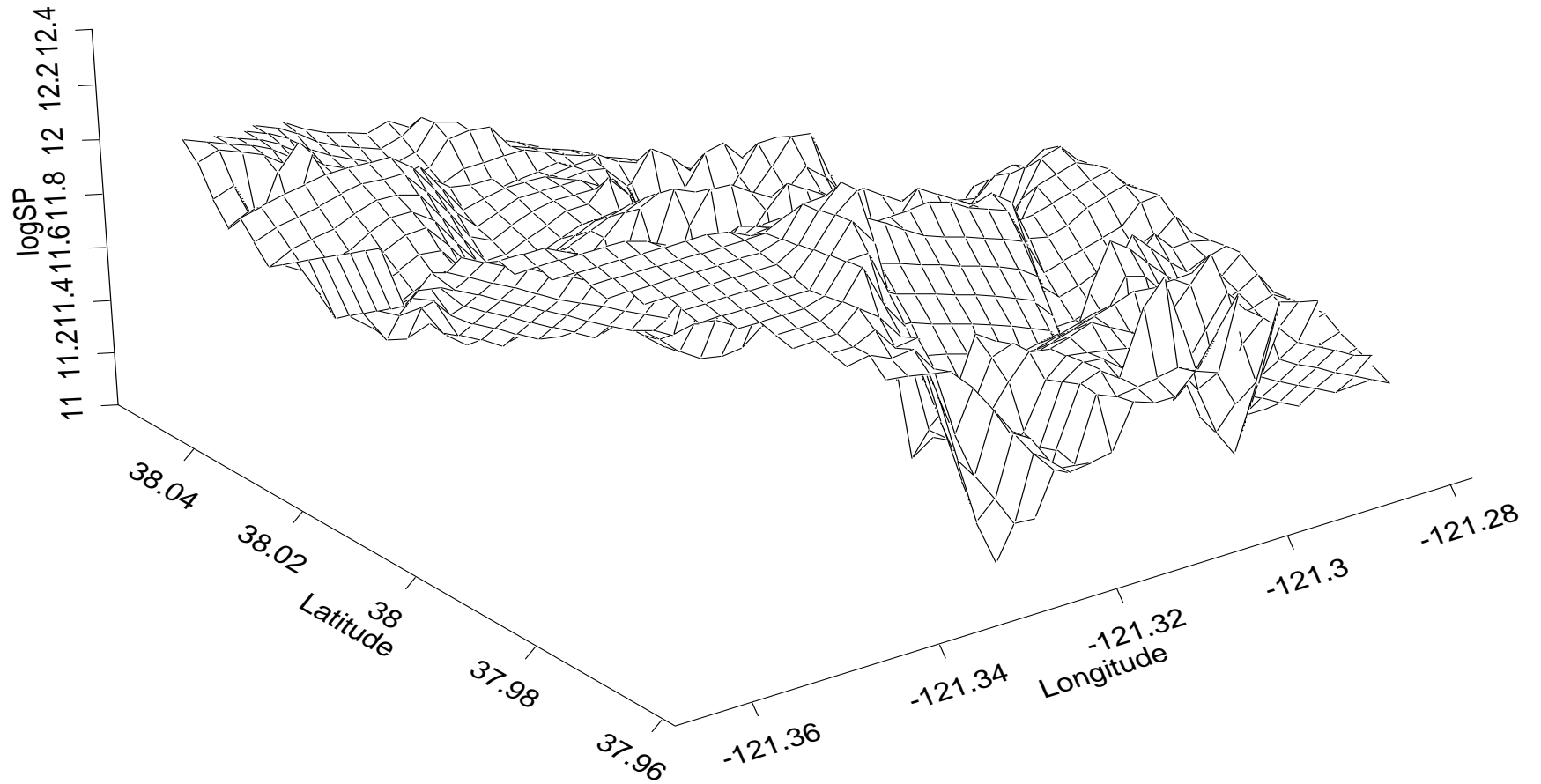
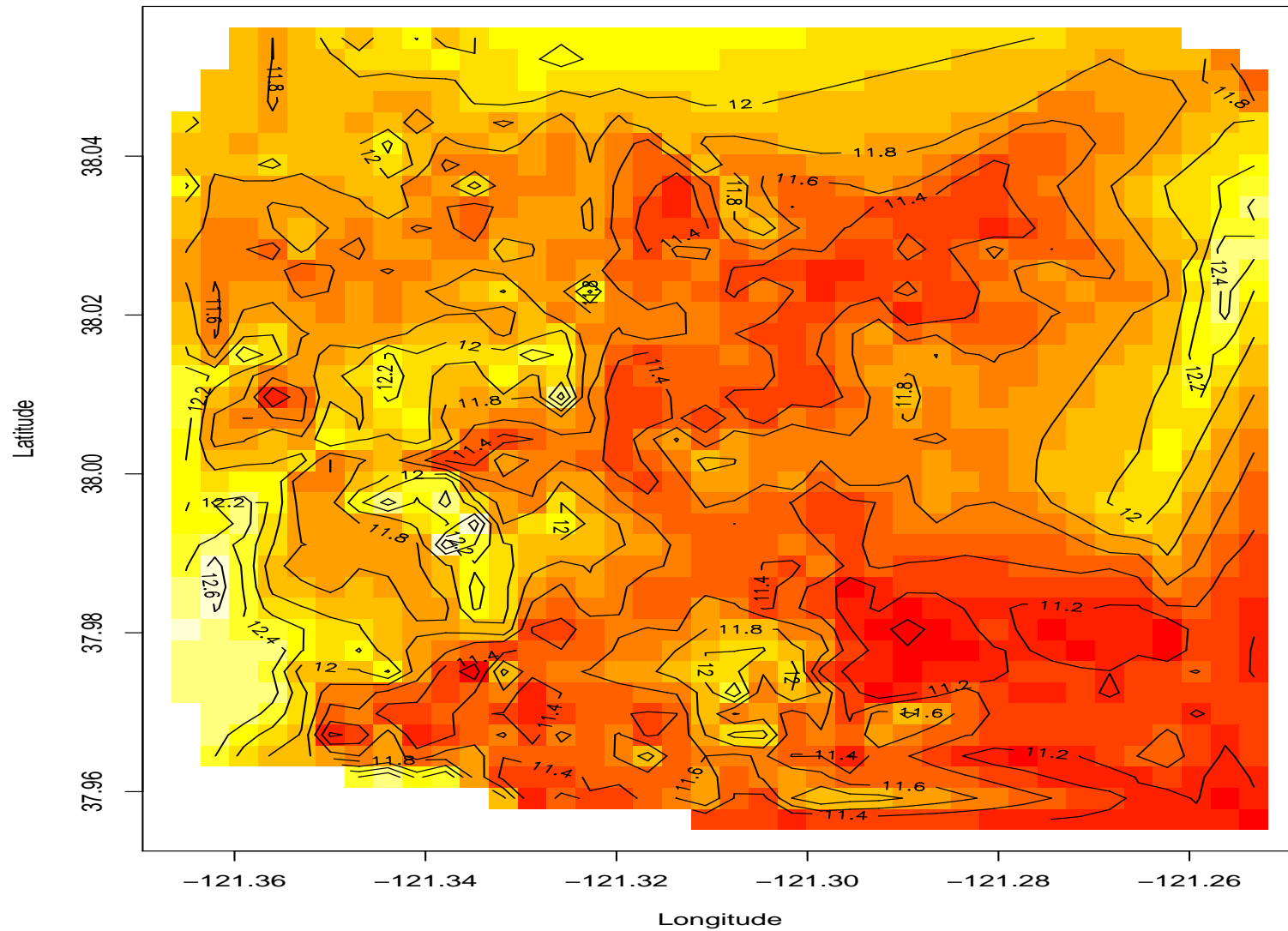
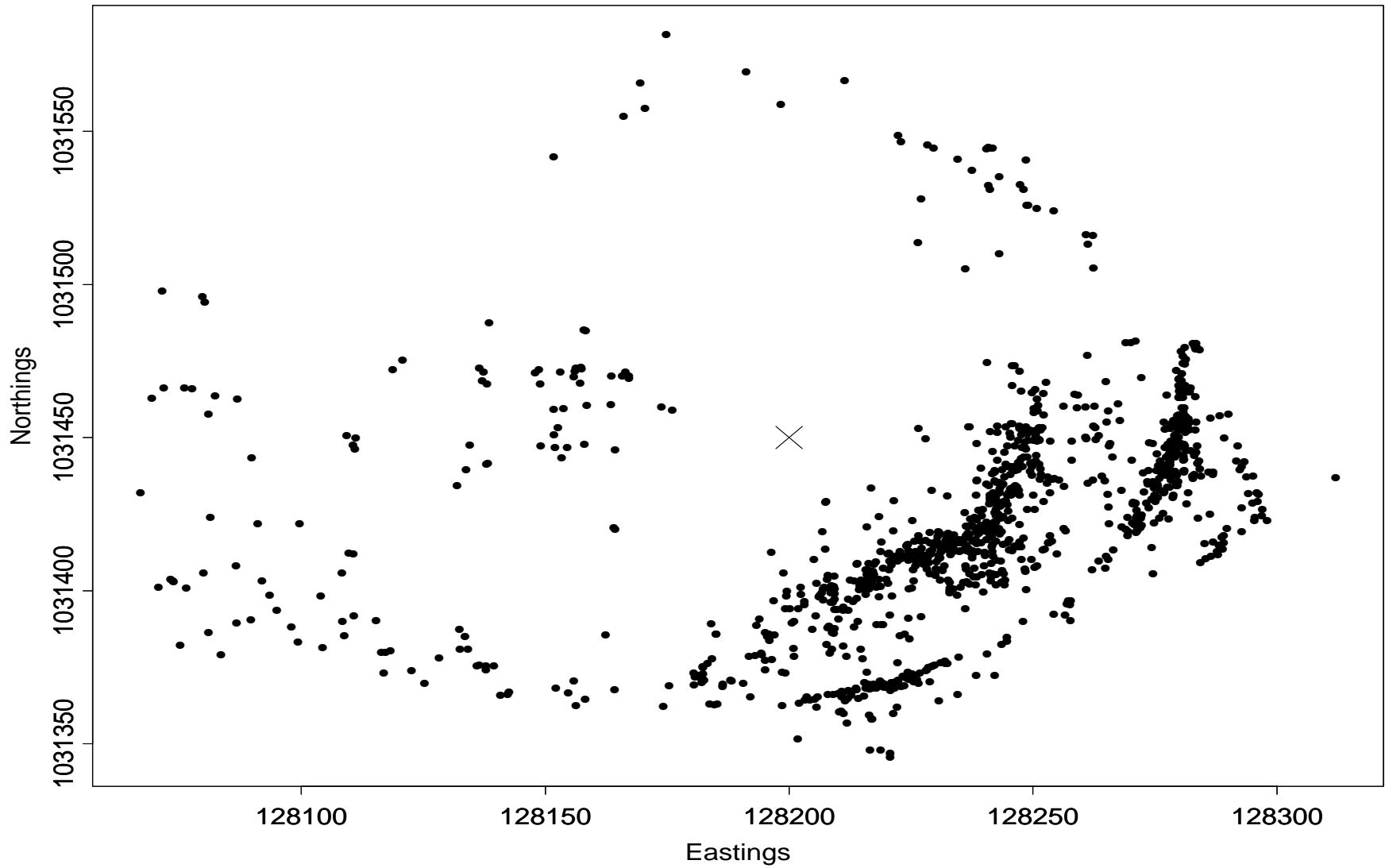


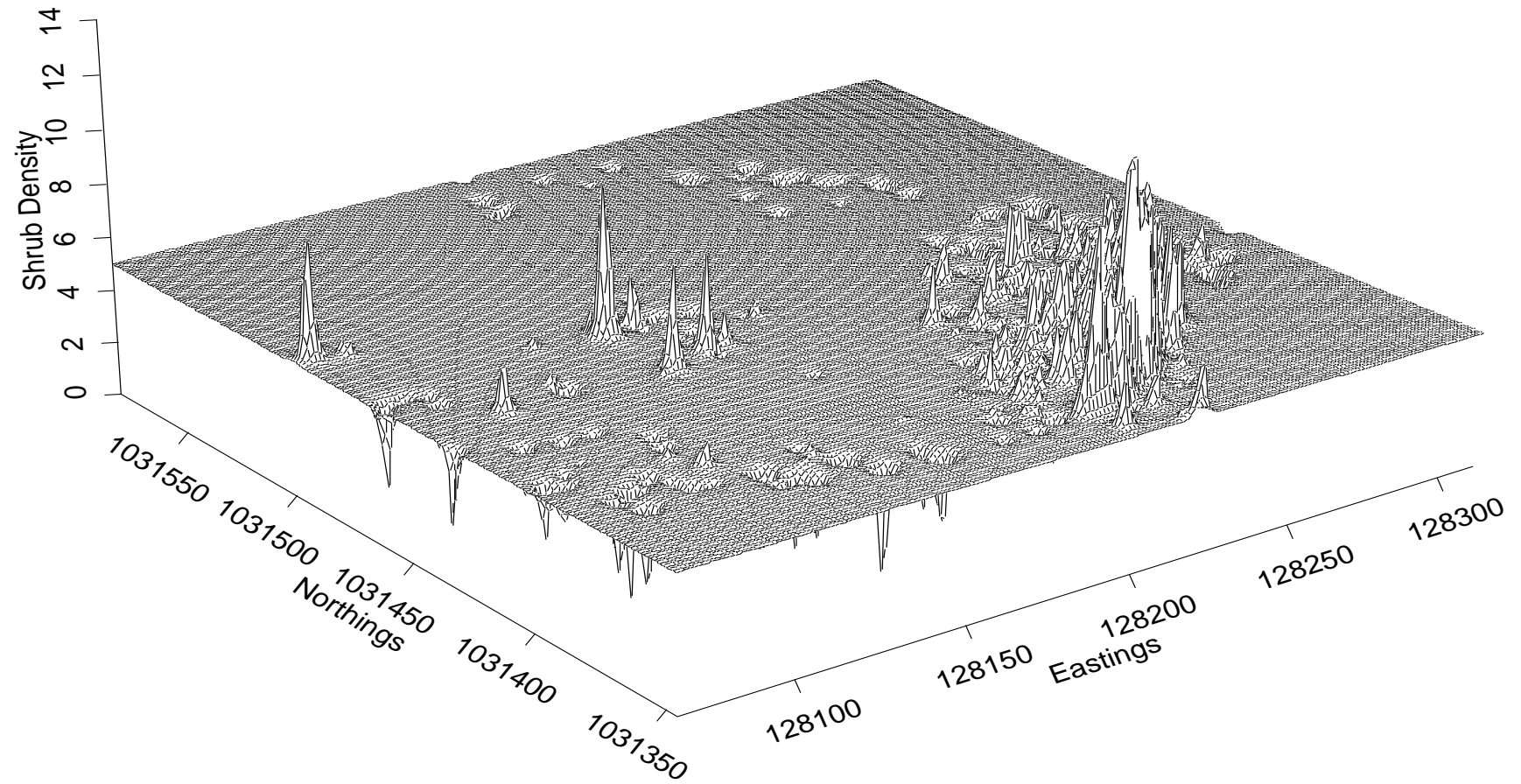
Image contour plot



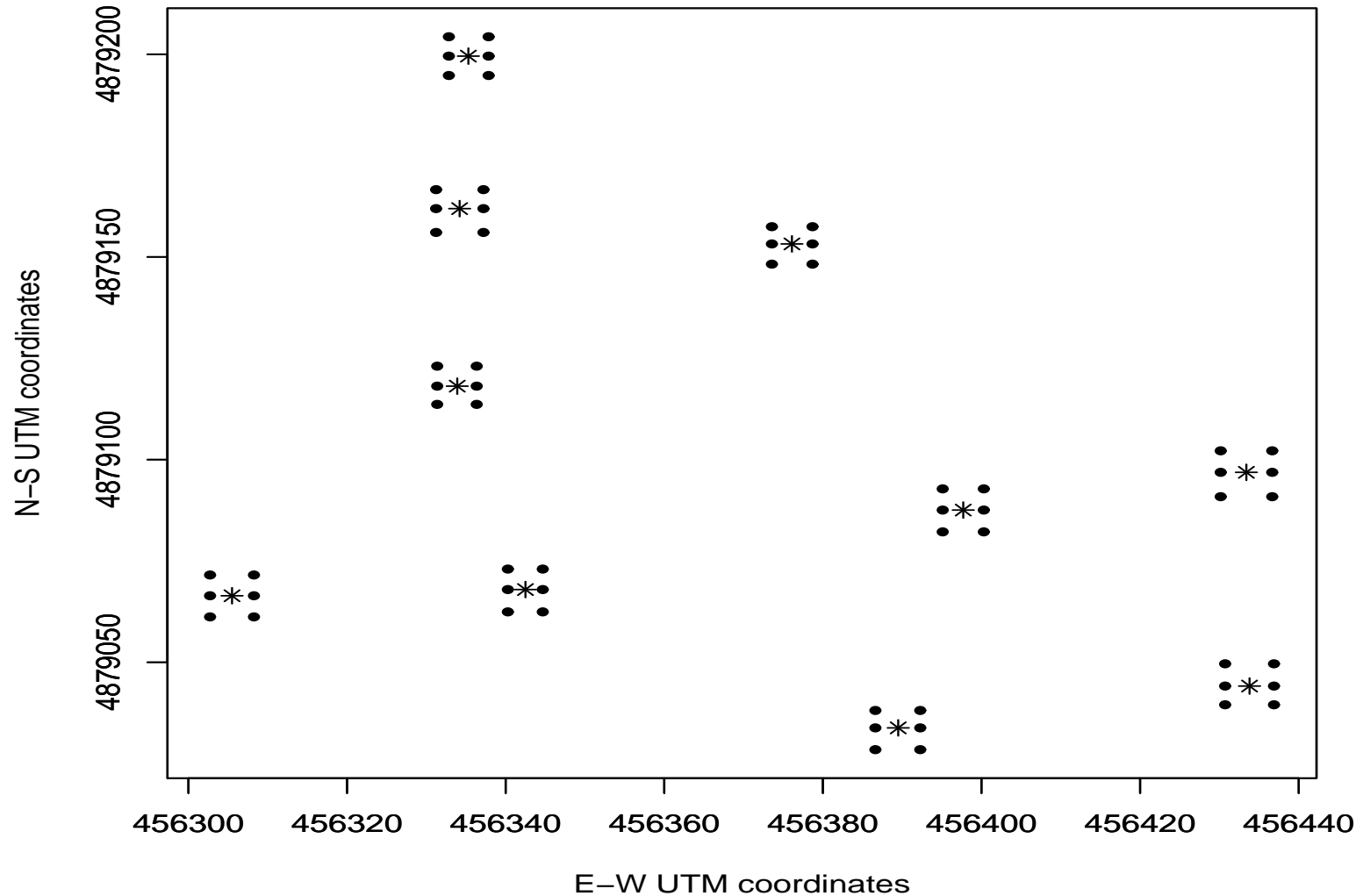
Locations form patterns



Surface features



Interesting plot arrangements



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- **Conceptually:** Pollutant level exists at all possible sites
- **Practically:** Data will be a partial realization of a spatial process – observed at $\{s_1, \dots, s_n\}$
- **Statistical objectives:** **Inference** about the process $Y(s)$; **predict** at new locations.

Stationary Gaussian Processes

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- **Weak stationarity:** Constant mean $\mu(\mathbf{s}) = \mu$, and $Cov(Y(\mathbf{s}), Y(\mathbf{s} + \mathbf{h})) = C(\mathbf{h})$: the covariance depends only upon the displacement (or separation) vector.

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- For Gaussian processes, strong and weak stationarity are equivalent.

Variograms

- Suppose we assume $E[Y(\mathbf{s} + \mathbf{h}) - Y(\mathbf{s})] = 0$ and define

$$E[Y(\mathbf{s} + \mathbf{h}) - Y(\mathbf{s})]^2 = Var(Y(\mathbf{s} + \mathbf{h}) - Y(\mathbf{s})) = 2\gamma(\mathbf{h}) .$$

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Note that intrinsic stationarity defines **only** the first and second moments of the differences $Y(\mathbf{s} + \mathbf{h}) - Y(\mathbf{s})$. It says nothing about the **joint** distribution of a collection of variables $Y(\mathbf{s}_1), \dots, Y(\mathbf{s}_n)$, and thus provides **no likelihood**.

Intrinsic Stationarity and Ergodicity

- Relationship between $\gamma(\mathbf{h})$ and $C(\mathbf{h})$:

$$\begin{aligned}2\gamma(\mathbf{h}) &= \text{Var}(Y(\mathbf{s} + \mathbf{h})) + \text{Var}(Y(\mathbf{s})) - 2\text{Cov}(Y(\mathbf{s} + \mathbf{h}), Y(\mathbf{s})) \\ &= C(\mathbf{0}) + C(\mathbf{0}) - 2C(\mathbf{h}) \\ &= 2[C(\mathbf{0}) - C(\mathbf{h})].\end{aligned}$$

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- So $\lim_{\|\mathbf{u}\| \rightarrow \infty} \gamma(\mathbf{u}) = C(\mathbf{0})$, and we can recover C from γ as long as this limit exists.

$$C(\mathbf{h}) = \lim_{\|\mathbf{u}\| \rightarrow \infty} \gamma(\mathbf{u}) - \gamma(\mathbf{h}).$$

Isotropy

- When $\gamma(\mathbf{h})$ or $C(\mathbf{h})$ depends upon the separation vector only through the distance $\|\mathbf{h}\|$, we say that the process is *isotropic*. In that case, we write $\gamma(\|\mathbf{h}\|)$ or $C(\|\mathbf{h}\|)$. Otherwise we say that the process is *anisotropic*.

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Isotropic processes are popular because of their *simplicity*, *interpretability*, and because a number of relatively *simple parametric forms* are available as candidates for C (and γ). Denoting $\|\mathbf{h}\|$ by t for notational simplicity, the next two tables provide a few examples...

Some common isotropic variograms

model	Variogram, $\gamma(t)$
Linear	$\gamma(t) = \begin{cases} \tau^2 + \sigma^2 t & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$
Spherical	$\gamma(t) = \begin{cases} \tau^2 + \sigma^2 & \text{if } t \geq 1/\phi \\ \tau^2 + \sigma^2 \left[\frac{3}{2}\phi t - \frac{1}{2}(\phi t)^3 \right] & \text{if } 0 < t < 1/\phi \\ 0 & \text{otherwise} \end{cases}$
Exponential	$\gamma(t) = \begin{cases} \tau^2 + \sigma^2(1 - \exp(-\phi t)) & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$
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Matérn at $\nu = 3/2$	$\gamma(t) = \begin{cases} \tau^2 + \sigma^2 \left[1 - (1 + \phi t) e^{-\phi t} \right] & \text{if } t > 0 \\ 0 & \text{o/w} \end{cases}$

Examples: Spherical Variogram

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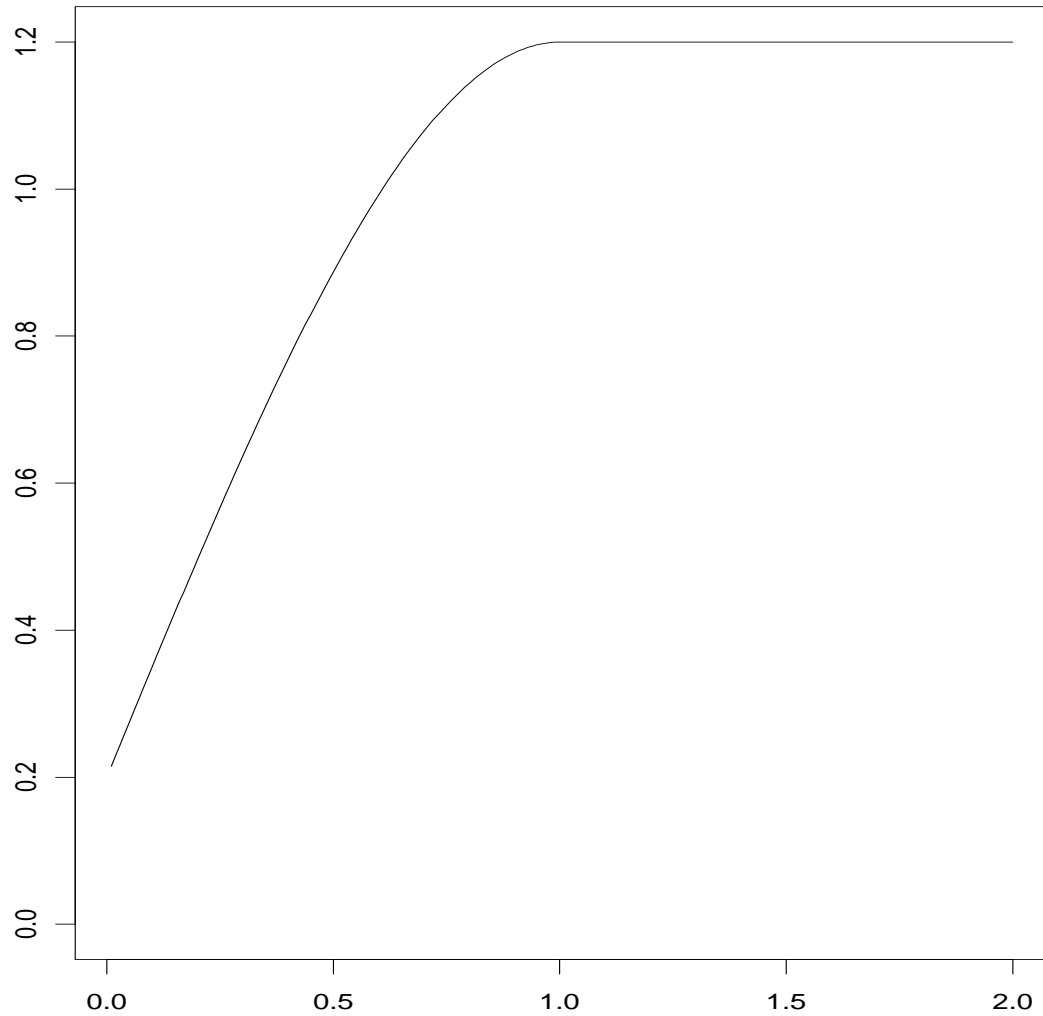
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- Finally, the value $t = 1/\phi$ at which $\gamma(t)$ first reaches its ultimate level (the sill) is called the *range*, $R \equiv 1/\phi$.



b) spherical; $a_0 = 0.2$, $a_1 = 1$, $R = 1$

Some common isotropic covariograms

Model	Covariance function, $C(t)$
Linear	$C(t)$ does not exist
Spherical	$C(t) = \begin{cases} 0 & \text{if } t \geq 1/\phi \\ \sigma^2 \left[1 - \frac{3}{2}\phi t + \frac{1}{2}(\phi t)^3 \right] & \text{if } 0 < t \leq 1/\phi \\ \tau^2 + \sigma^2 & \text{otherwise} \end{cases}$
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- We define the *effective range*, t_0 , as the distance at which this correlation has dropped to only 0.05. Setting $\exp(-\phi t_0)$ equal to this value we obtain $t_0 \approx 3/\phi$, since $\log(0.05) \approx -3$.

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- Finally, the form of $C(t)$ shows why the nugget τ^2 is often viewed as a “nonspatial effect variance,” and the partial sill (σ^2) is viewed as a “spatial effect variance.”

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- ν is a smoothness parameter (a *fractal*) controlling process smoothness

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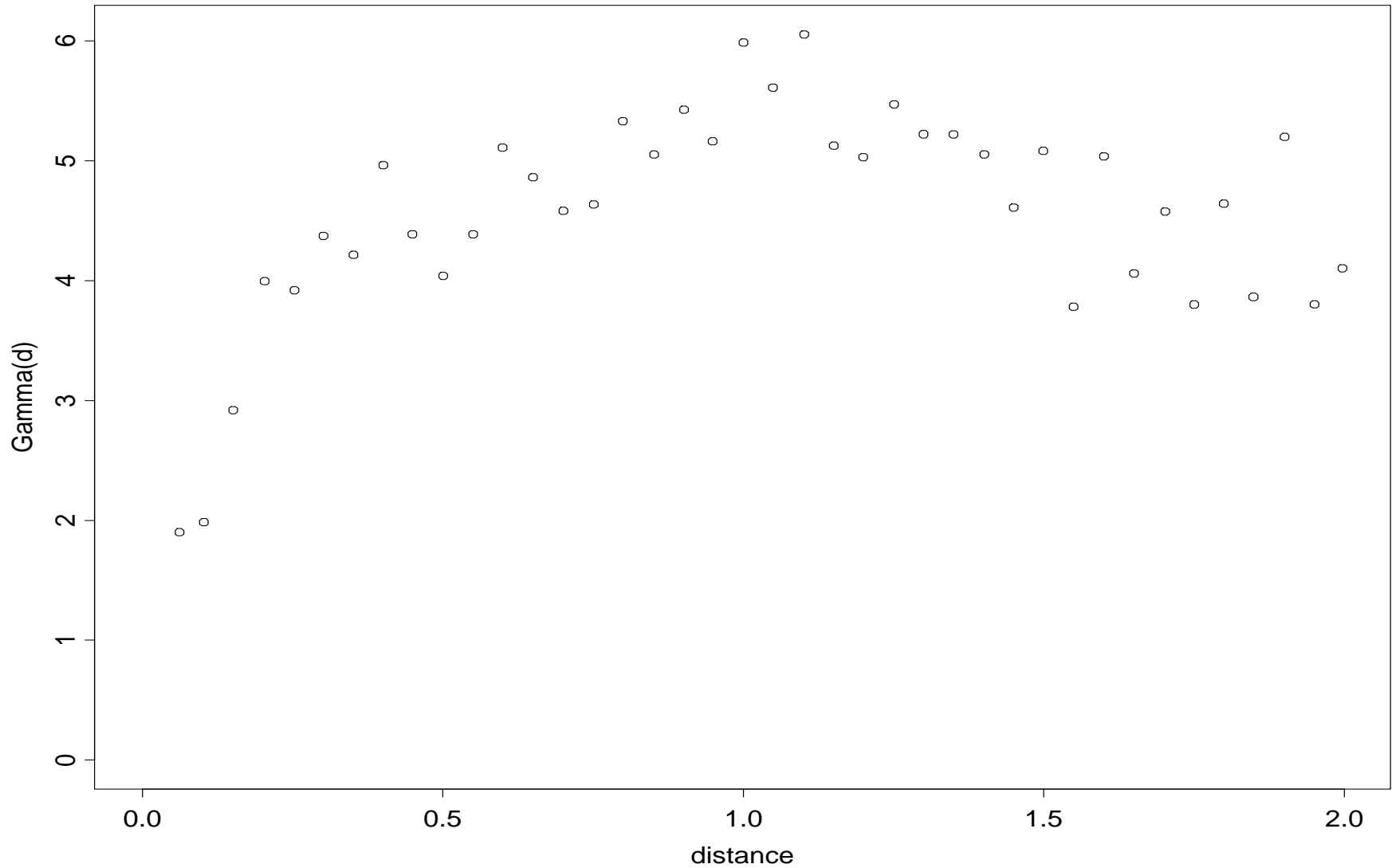
$$\gamma(t) = \frac{1}{2|N(t)|} \sum_{\mathbf{s}_i, \mathbf{s}_j \in N(t)} (Y(\mathbf{s}_i) - Y(\mathbf{s}_j))^2$$

where $N(t)$ is the number of points such that $\|\mathbf{s}_i - \mathbf{s}_j\| = t$ and $|N(t)|$ is the number of points in $N(t)$.

- Grid up the t space into intervals $I_1 = (0, t_1)$, $I_2 = (t_1, t_2)$, and so forth, up to $I_K = (t_{K-1}, t_K)$. Representing t values in each interval by its midpoint, we define:

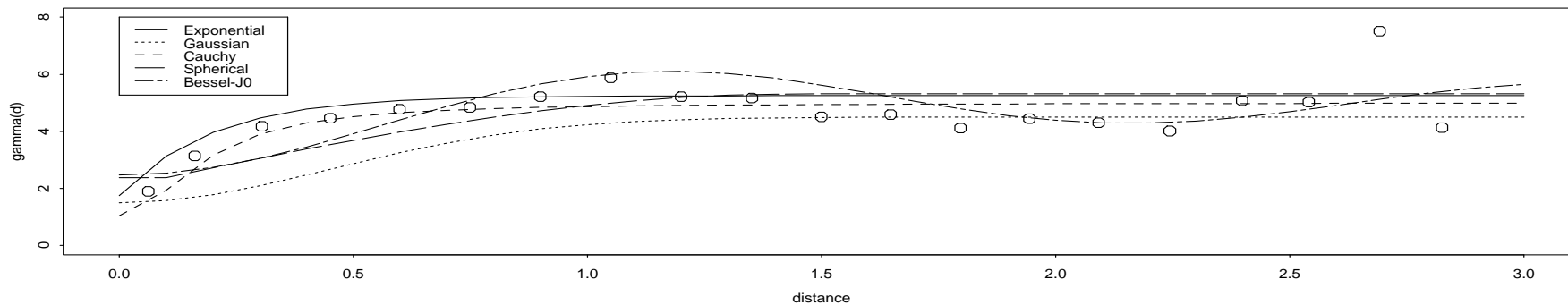
$$N(t_k) = \{(\mathbf{s}_i, \mathbf{s}_j) : \|\mathbf{s}_i - \mathbf{s}_j\| \in I_k\}, k = 1, \dots, K.$$

Empirical variogram: scallops data

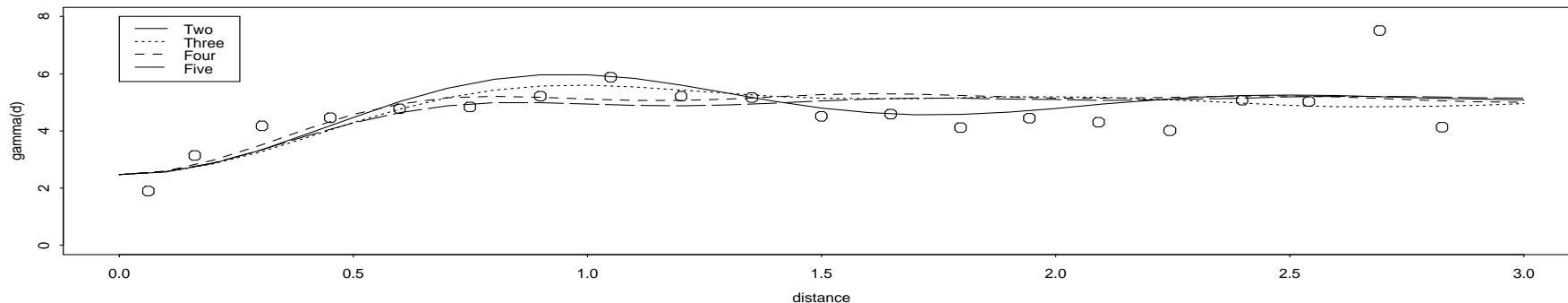


Empirical variogram: scallops data

Parametric Semivariograms



Bessel Mixtures - Random Weights



Bessel Mixtures - Random Phi's

