Multivariate Lattice Models for Areal Data

with Application to Multiple Disease Mapping

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Outline

- Background and motivation for disease mapping
- Overview of MCAR models in the literature
- Multivariate lattice models with its application
  - Generalized multivariate CAR (GMCAR) models
  - Order-free coregionalized lattice models
- Ongoing research

Background: disease mapping

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  - to describe the geographic variation of disease
  - to generate hypotheses about the possible causes for differences in risk of disease.

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- Related spatial database:
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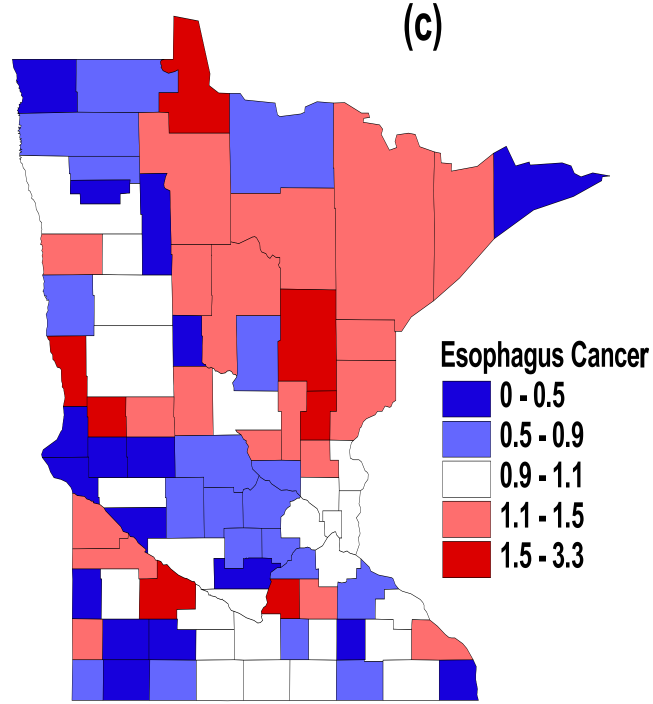
  - Surveillance, Epidemiology, and End Results (SEER)
- Standard mortality ratio: $SMR = Y_i/E_i$
  - Y_i is the observed number of deaths in region i.
  - E_i is the expected number of deaths in region i.

Background: single disease mapping

Map of raw standard mortality ratios (SMR) of esophagus cancer between 1991 and 1998 in Minnesota counties



Background: modelling of a single disease

For rare disease, poisson regression model:

$$Y_i \mid \mu_i \overset{ind}{\sim} Poisson(E_i \exp(\mu_i)) \; i = 1, \ldots, n,$$

where $\mu_i = \mathbf{x}_i' \boldsymbol{\beta} + \phi_i$. The $\mathbf{x}_i$ are explanatory, region-level spatial covariates, having parameter coefficients $\boldsymbol{\beta}$.

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- $E(Y_i) = E_i \exp(\mu_i) \Longrightarrow \hat{S}MR = \exp(\hat{\mu}_i)$, where $\hat{\mu}_i = \mathbf{x}_i' \hat{\boldsymbol{\beta}} + \hat{\phi}_i$
- μ_i represents the log-relative risk of departures of the Y_i from the E_i.

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- Hierarchical Bayesian modeling:
  - Using Markov chain Monte Carlo (MCMC) methods
  - First stage: likelihood for observation data
  - Second stage: prior distributions for fixed effects $\boldsymbol{\beta}$ and random effects $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)'$

Background: modeling of a single disease (Cont'd)

- Markov random field (MRF): the conditional distribution of a site's response given the responses of all the other sites depends only on the observations in the neighborhood of this site.

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- Conditionally autoregressive (CAR) prior on $\phi = (\phi_1, \ldots, \phi_n)'$

$$\phi_i \mid \phi_j, j \neq i, \sim N\left(\frac{\alpha}{m_i}\sum_{i\sim j}\phi_j, \ \frac{1}{\tau m_i}\right), \ i, j = 1, \ldots, n,$$

  where m_i is the number of neighbors of area i and α is *smoothing* parameter.

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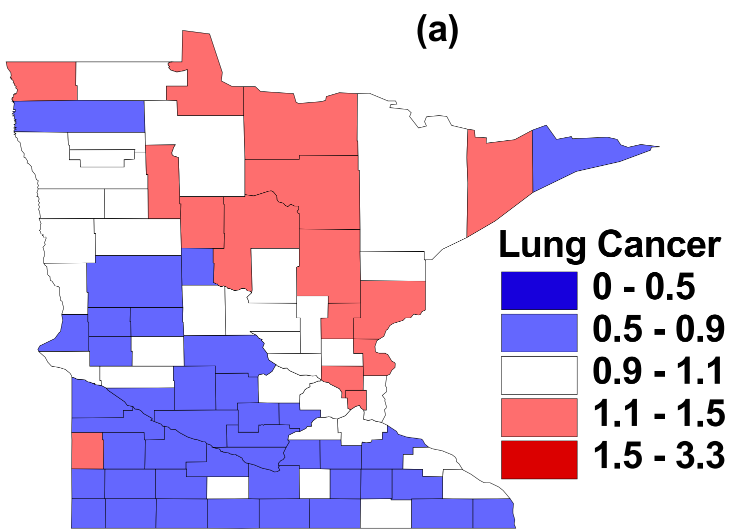
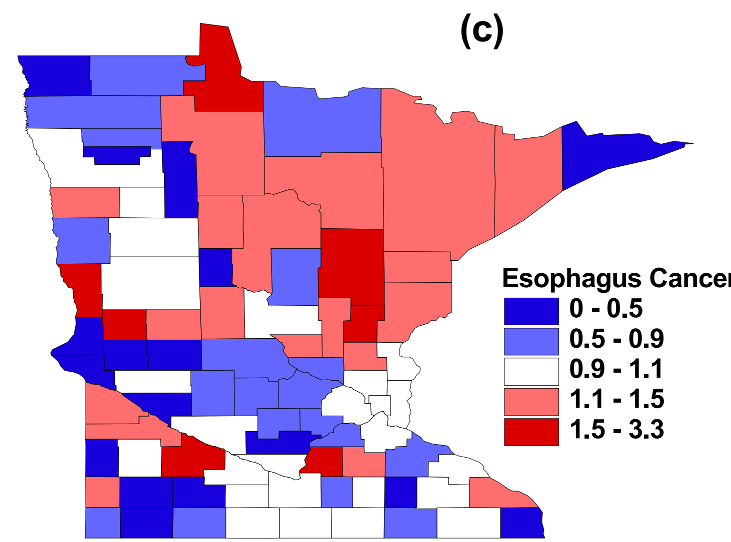
$$\Longleftrightarrow \quad \boldsymbol{\phi} \;\sim\; N_n\left(\mathbf{0}\,,\; \left[\tau\left(D-\alpha W\right)\right]^{-1}\right),$$

  where $D = Diag(m_i)$, and W is the adjacency matrix of the map (i.e., $w_{ii} = 0$, and $w_{ii'} = 1$ if i' is adjacent to i and 0 otherwise).

Motivation: multiple disease mapping

Maps of raw standard mortality ratios (SMR) of lung and esophagus cancer between 1991 and 1998 in Minnesota counties





Motivation: spatial modeling of multiple diseases

- For rare diseases, poisson regression model:

$$Y_{ij} \overset{ind}{\sim} Poisson(E_{ij} e^{\mathbf{x}'_{ij}\boldsymbol{\beta}_j + \phi_{ij}}) \,, \; i = 1, \ldots, n, \; j = 1, \ldots, p,$$

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- Correlations in multiple disease data:
  - Spatial correlation for each disease across regions
  - Dependence among multiple diseases within the same region
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– To develop multivariate lattice models for mapping multiple disease

Simple MCAR model: $MCAR(\alpha,\, \Lambda)$

Hierarchical model: $Y_{ij} \stackrel{ind}{\sim} f(y_{ij} \mid \boldsymbol{\beta}, \phi_{ij}),\ i = 1, \ldots n\,, j = 1, \ldots p\,.$
$MCAR(\alpha,\, \Lambda)$: generalized from the univariate CAR under the "separability" assumption (Carlin and Banerjee, 2003; Gelfand and Vounatsou, 2003.)

$$\boldsymbol{\phi} \ \sim N_{np}\left(\mathbf{0},\ \Lambda^{-1} \otimes (D - \alpha W)^{-1}\right)\ ,$$

where $\boldsymbol{\phi} = (\boldsymbol{\phi}_1', \ldots, \boldsymbol{\phi}_p')'$, $\boldsymbol{\phi}_k = (\phi_{1k}, \ldots, \phi_{nk})'$, $k = 1, \ldots p$, Λ^{-1} is a $p \times p$ positive definite matrix, $D = Diag(m_i)$, m_i is the number of neighbors of region i, W is the adjacency matrix of the map (i.e., $w_{ii} = 0$, and $w_{ii'} = 1$ if i' is adjacent to i and 0 otherwise), and α is called *smoothing* parameter.

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- $MCAR(1, \Lambda)$: $\alpha = 1$, an improper MCAR distribution (MIAR).
- $MCAR(\alpha, \Lambda)$: a proper MCAR distribution with $|\alpha| < 1$.

$MCAR(\alpha_1, \ldots, \alpha_p,\ \Lambda)$ model

$Y_{ij} \overset{ind}{\sim} f(y_{ij} \mid \boldsymbol{\beta}, \phi_{ij})$, $\boldsymbol{\phi} \sim N_{2n}(0,\ \Sigma)$, $i = 1, \ldots n$, $j = 1, 2$. ($p = 2$)

Simple $MCAR(\alpha, \Lambda)$ model: $\Sigma^{-1} = \begin{pmatrix} (D - \alpha W)\Lambda_{11} & (D - \alpha W)\Lambda_{12} \\ (D - \alpha W)\Lambda_{12} & (D - \alpha W)\Lambda_{22} \end{pmatrix}$

$MCAR(\alpha_1, \ldots, \alpha_p,\ \Lambda)$ model:

$$\Sigma^{-1} = \begin{pmatrix} R_1' & 0 \\ 0 & R_2' \end{pmatrix} (\Lambda \otimes I_{n \times n}) \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} = \begin{pmatrix} R_1' R_1 \Lambda_{11} & R_1' R_2 \Lambda_{12} \\ R_2' R_1 \Lambda_{12} & R_2' R_2 \Lambda_{22} \end{pmatrix},$$

where $R_j' R_j = D - \alpha_j W$, $|\alpha_j| < 1$, $j = 1, 2$.

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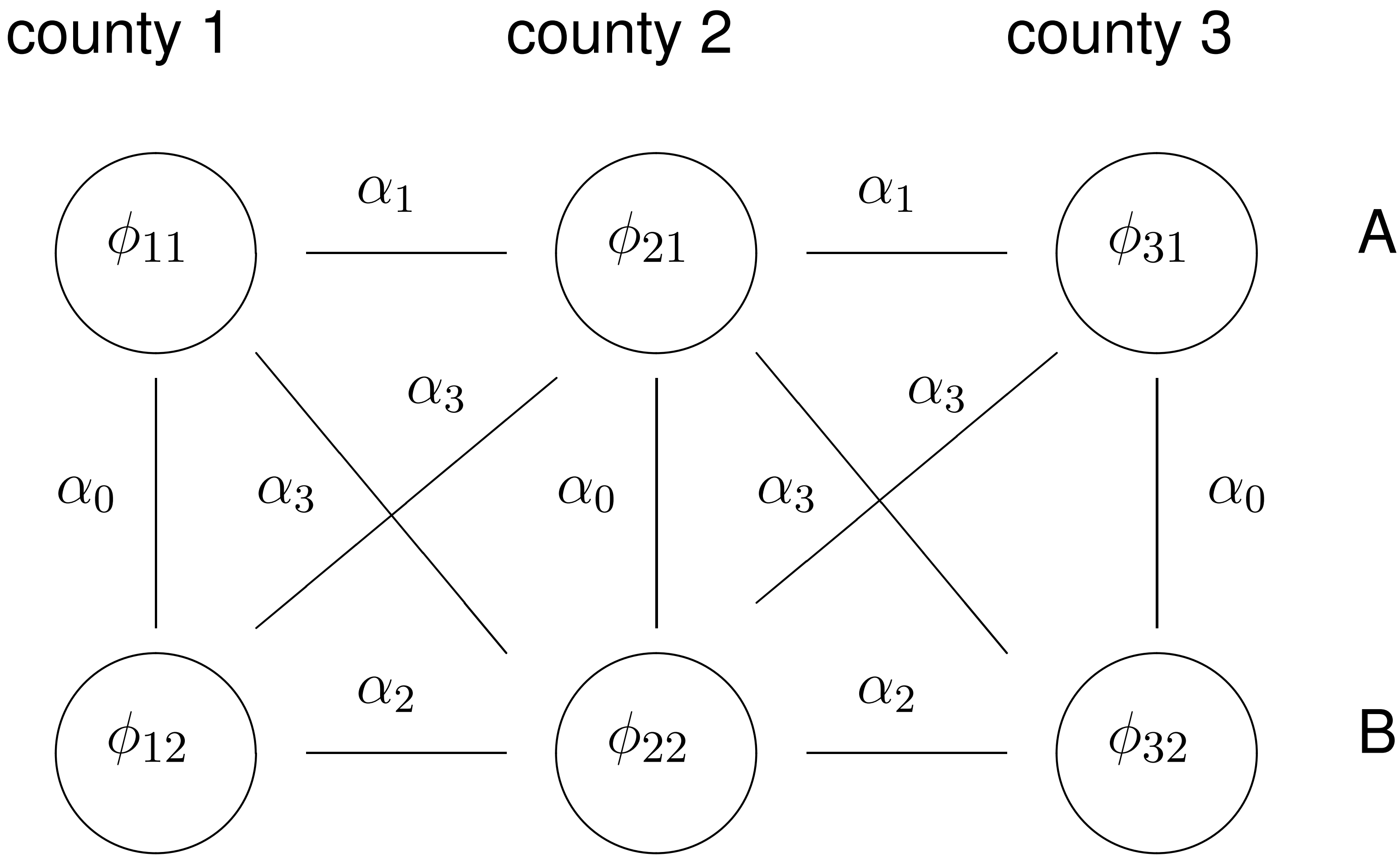
where $R_j' R_j = D - \alpha_j W\ ,\ |\alpha_j| < 1\ ,\ j = 1, 2.$

- R_j as the Cholesky decomposition of $D - \alpha_j W$ (Carlin and Banerjee, 2003).
- R_j as the spectral decomposition of $D - \alpha_j W$ (Gelfand and Vounatsou, 2003).

General MCAR model



Kim's twofold CAR model

Hierarchical model: $Y_{ij} \overset{ind}{\sim} f(y_{ij} \mid \boldsymbol{\beta}, \phi_{ij}), i = 1, \ldots n\ , j = 1, 2.$ ($p = 2$)
Conditional distribution is a Gaussian with

- $E(\phi_{ij}|\phi_{il}, \phi_{kj}, \phi_{kl}) = \frac{1}{2m_i+1}\left(\alpha_j \sum_{k\sim i} \phi_{kj} + \alpha_3 \sqrt{\frac{\tau_l}{\tau_j}} \sum_{k\sim i} \phi_{kl} + \alpha_0 \sqrt{\frac{\tau_l}{\tau_j}} \phi_{il}\right)$
- $Var(\phi_{ij}|\phi_{il}, \phi_{kj}, \phi_{kl}) = \frac{\tau_j^{-1}}{2m_i+1}$ for $i, k = 1, \ldots, n\ , j, l = 1, 2,\ l \neq j$.

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$\Longrightarrow$ Joint distribution $\boldsymbol{\phi} \sim N_{2n}(0, \ \Sigma)$,

$$\Sigma^{-1} = \begin{pmatrix} (2D + I - \alpha_1 W)\tau_1 & -(\alpha_0 I + \alpha_3 W)\sqrt{\tau_1 \tau_2} \\ -(\alpha_0 I + \alpha_3 W)\sqrt{\tau_1 \tau_2} & (2D + I - \alpha_2 W)\tau_2 \end{pmatrix}$$

conditions for positive definiteness: $|\alpha_l| < 1,\ l = 0, 1, 2, 3$

Conditional distribution $\phi_1|\phi_2$

$\boldsymbol{\phi}_1 = (\phi_{11}, \ldots, \phi_{n1})'$, $\boldsymbol{\phi}_2 = (\phi_{12}, \ldots, \phi_{n2})'$,

- $MCAR(\alpha,\ \Lambda)$ model: $E(\boldsymbol{\phi}_1|\boldsymbol{\phi}_2) = -\frac{\Lambda_{12}}{\Lambda_{11}}\boldsymbol{\phi}_2$
- $MCAR(\alpha_1, \alpha_2,\ \Lambda)$ model:
  $E(\boldsymbol{\phi}_1|\boldsymbol{\phi}_2) = -\frac{\Lambda_{12}}{\Lambda_{11}}(D - \alpha_1 W)^{-\frac{1}{2}}(D - \alpha_2 W)^{\frac{1}{2}}\boldsymbol{\phi}_2$
- Kim's twofold CAR $2fCAR(\alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \tau_1,\ \tau_2)$ model:
  $E(\boldsymbol{\phi}_1|\boldsymbol{\phi}_2) = \sqrt{\frac{\tau_2}{\tau_1}}(2D + I - \alpha_1 W)^{-1}(\alpha_0 I + \alpha_3 W)\boldsymbol{\phi}_2$

in all cases, $Var(\boldsymbol{\phi}_1|\boldsymbol{\phi}_2) = [\Lambda_{11}(D - \alpha_1 W)]^{-1}$ or $[\tau_1(D - \alpha_1 W)]^{-1}$

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Now we can assume:

- $E(\boldsymbol{\phi}_1|\boldsymbol{\phi}_2) = (\eta_0 I + \eta_1 W)\boldsymbol{\phi}_2$
- $Var(\boldsymbol{\phi}_1|\boldsymbol{\phi}_2) = [\tau_1(D - \alpha_1 W)]^{-1}$

GMCAR model

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Specify:

$$\boldsymbol{\phi}_1 | \boldsymbol{\phi}_2 \;\sim\; N\left((\eta_0 I + \eta_1 W)\boldsymbol{\phi}_2 \,,\; [\tau_1(D - \alpha_1 W)]^{-1}\right)$$

$$\boldsymbol{\phi}_2 \;\sim\; N\left(\mathbf{0} \,,\; [\tau_2(D - \alpha_2 W)]^{-1}\right)$$

The joint distribution: $p(\boldsymbol{\phi}) = p(\boldsymbol{\phi}_1 | \boldsymbol{\phi}_2) p(\boldsymbol{\phi}_2)$

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- Implemented in the popular `WinBUGS` package!
- Link between GMCAR and MCAR models:
  - Simple MCAR model: $\alpha_1 = \alpha_2 = \alpha$, $\eta_1 = 0$
  - One special $MCAR(\alpha_1, \alpha_2)$ model: $\eta_1 = 0$, $\alpha_1 \neq \alpha_2$
  - GMCAR model: α_1, α_2, τ_1, τ_2, η_0, η_1 (same parameters as Kim's twofold CAR)

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    - With three variables the ordering produces six competing models.
    - With more than three variables ...

Linear model of coregionalization (LMC)

Hierarchical model: $Y_{ij} \stackrel{ind}{\sim} f(y_{ij} \mid \boldsymbol{\beta}, \phi_{ij}),\ \ i = 1, \ldots n\,, j = 1, \ldots p.$
Let $\boldsymbol{\phi} = (\boldsymbol{\psi}'_1, \ldots, \boldsymbol{\psi}'_n)'$, where $\boldsymbol{\psi}_i = (\phi_{i1}, \ldots, \phi_{ip})'$, $i = 1, \ldots, n$.

- For region i, assuming $\boldsymbol{\psi}_i = A\mathbf{v}_i$, where $\mathbf{v}_i = (v_{i1}, \ldots, v_{ip})'$, and A is a $p \times p$ linear transformation matrix.
  $\Longrightarrow$ random effects $\boldsymbol{\phi} = (I_{n\times n} \otimes A)\mathbf{v}$, where $\mathbf{v} = (\mathbf{v}'_1, \ldots, \mathbf{v}'_n)'$ and and I is a $n \times n$ identity matrix.

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- Rearranging $\boldsymbol{\phi} = (\boldsymbol{\psi}_1', \ldots, \boldsymbol{\psi}_n')'$ as $\boldsymbol{\phi} = (\boldsymbol{\phi}_1', \ldots, \boldsymbol{\phi}_p')'$, $\boldsymbol{\phi}_j = (\phi_{1j}, \ldots, \phi_{nj})'$.
$\Longrightarrow$ random effects $\boldsymbol{\phi} = (A \otimes I_{n \times n})\mathbf{u}$, where $\mathbf{u} = (\mathbf{u}_1', \ldots, \mathbf{u}_p')'$ is the corresponding permutation of $\mathbf{v}$ with $\mathbf{u}_j = (v_{1j}, \ldots, v_{nj})',\ j = 1, \ldots, p$.

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  $\Longrightarrow$ random effects $\boldsymbol{\phi} = (I_{n\times n} \otimes A)\mathbf{v}$, where $\mathbf{v} = (\mathbf{v}'_1, \ldots, \mathbf{v}'_n)'$ and and I is a $n \times n$ identity matrix.
- Rearranging $\boldsymbol{\phi} = (\boldsymbol{\psi}'_1, \ldots, \boldsymbol{\psi}'_n)'$ as $\boldsymbol{\phi} = (\boldsymbol{\phi}'_1, \ldots, \boldsymbol{\phi}'_p)'$, $\boldsymbol{\phi}_j = (\phi_{1j}, \ldots, \phi_{nj})'$.
  $\Longrightarrow$ random effects $\boldsymbol{\phi} = (A \otimes I_{n\times n})\mathbf{u}$, where $\mathbf{u} = (\mathbf{u}'_1, \ldots, \mathbf{u}'_p)'$ is the corresponding permutation of $\mathbf{v}$ with $\mathbf{u}_j = (v_{1j}, \ldots, v_{nj})',\ j = 1, \ldots, p$.
- Each component $\mathbf{u}_j$ of $\mathbf{u}$ is an n-dimensional latent spatial process.

Linear model of coregionalization (LMC)

Hierarchical model: $Y_{ij} \stackrel{ind}{\sim} f(y_{ij} \mid \boldsymbol{\beta}, \phi_{ij}),\ \ i = 1, \ldots n\,, j = 1, \ldots p.$
Let $\boldsymbol{\phi} = (\boldsymbol{\psi}_1', \ldots, \boldsymbol{\psi}_n')'$, where $\boldsymbol{\psi}_i = (\phi_{i1}, \ldots, \phi_{ip})'$, $i = 1, \ldots, n$.

- For region i, assuming $\boldsymbol{\psi}_i = A\mathbf{v}_i$, where $\mathbf{v}_i = (v_{i1}, \ldots, v_{ip})'$, and A is a $p \times p$ linear transformation matrix.
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- Each component $\mathbf{u}_j$ of $\mathbf{u}$ is an n-dimensional latent spatial process.
- Obtain different multivariate lattice models by making different (simpler) assumptions about the p latent spatial (areal) processes $\mathbf{u}_j$.

Case 1: p independent and identical spatial processes

- Assuming that the latent spatial processes $\mathbf{u}_j$ are independent and identical ($j = 1, \ldots, p$).
- Spatial processes $\mathbf{u}_j \sim N_n\left(\mathbf{0},\, (D - \alpha W)^{-1}\right),\, j = 1, \ldots, p.$

$\Longrightarrow \mathbf{u} \sim N_{np}\left(\mathbf{0},\, I_{p \times p} \otimes (D - \alpha W)^{-1}\right)$, where $\mathbf{u} = (\mathbf{u}_1', \ldots, \mathbf{u}_p')'$.
Since $\boldsymbol{\phi} = (A \otimes I_{n \times n})\mathbf{u}$, defining $\Sigma = AA'$, the joint distribution of $\boldsymbol{\phi}$ is

$$\boldsymbol{\phi} \sim N_{np}\left(\mathbf{0},\, \Sigma \otimes (D - \alpha W)^{-1}\right).$$

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- It can be implemented in `WinBUGS` by writing $\boldsymbol{\phi} = (A \otimes I_{n\times n})\mathbf{u}$ and assigning proper CAR priors (via the `car.proper` function) for each $\mathbf{u}_j,\; j = 1, \ldots, p$.

Case 2: p independent and not identical spatial processes

- Assuming that the latent spatial processes $\mathbf{u}_j$ are independent and not identical ($j = 1, \ldots, p$).
- Spatial processes $\mathbf{u}_j \sim N_n \left(\mathbf{0},\, (D - \alpha_j W)^{-1} \right),\ j = 1, \ldots, p.$

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Since $\boldsymbol{\phi} = (A \otimes I_{n \times n})\mathbf{u}$, the joint distribution of $\boldsymbol{\phi}$ is

$$\boldsymbol{\phi} \sim N_{np}\left(\mathbf{0},\ (A \otimes I_{n \times n})\Gamma^{-1}(A \otimes I_{n \times n})'\right),$$

where $\Sigma = AA'$ and Γ is an $np \times np$ block diagonal matrix with diagonal entries $\Gamma_j = D - \alpha_j W$, $j = 1, \ldots, p$.

- It is denoted as $MCAR(\alpha_1, \ldots, \alpha_p,\ \Sigma)$ distribution.

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- It is denoted as $MCAR(\alpha_1, \ldots, \alpha_p,\, \Sigma)$ distribution.
- The distribution is not invariant to the choice of A: since $APP'A' = AA' = \Sigma$, where P is an arbitrary orthogonal matrix, the matrix A is not identified.
- It is similar to the $MCAR(\alpha_1, \ldots, \alpha_p,\, \Lambda)$ structure.

Case 3: p dependent and not identical spatial processes

- Since $\boldsymbol{\psi}_i = A\mathbf{v}_i$, where $\boldsymbol{\psi}_i = (\phi_{i1}, \ldots, \phi_{ip})'$, and $\mathbf{v}_i = (v_{i1}, \ldots, v_{ip})'$, we can assume that v_{ij} and $v_{i,\,l\neq j}$ are independent given $v_{k\neq i,\,j}$ and $v_{k\neq i,\,l\neq j}$, where $l, j = 1, \ldots, p$ and $i, k = 1, \ldots, n$.
- Specify that the ij-th conditional distribution as Gaussian with conditional mean and conditional variance:

$$E(v_{ij}\,|v_{k\neq i,\,j},\; v_{i,\,l\neq j},\; v_{k\neq i,\,l\neq j}) = b_{jj}\left(\sum_{k\sim i}\frac{v_{kj}}{m_i}\right) + \sum_{l\neq j}\left[b_{jl}\left(\sum_{k\sim i}\frac{v_{kl}}{m_i}\right)\right],$$

$Var(v_{ij}\,|v_{k\neq i,\,j},\; v_{i,\,l\neq j},\; v_{k\neq i,\,l\neq j}) = \frac{1}{m_i}$, where b_{jj} denotes spatial autocorrelation for the random spatial process $\mathbf{u}_j$, while b_{jl} $(l \neq j,\; l, j = 1, \ldots, p)$ denotes cross-spatial correlation between the random spatial process $\mathbf{u}_j$ and $\mathbf{u}_l$.

Case 3: p dependent and not identical spatial processes (Cont'd)

- $\mathbf{u} \sim N_{np}\left(\mathbf{0}, (I_{p\times p} \otimes D - B \otimes W)^{-1}\right)$, where I is a $p \times p$ identity matrix and B is a $p \times p$ symmetric matrix with the elements b_{jl}.
- Since $\boldsymbol{\phi} = (A \otimes I_{n\times n})\mathbf{u}$, defining $\Sigma = AA'$, the joint distribution is

$$\boldsymbol{\phi} \sim N_{np}\left(\mathbf{0},\ (A \otimes I_{n\times n})\,(I_{p\times p} \otimes D - B \otimes W)^{-1}\,(A \otimes I_{n\times n})'\right).$$

  - It is denoted as the $MCAR(B, \Sigma)$ distribution.

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  - It is denoted as the $MCAR(B, \Sigma)$ distribution.
  - It is reduced to the $MCAR(\alpha_1, \ldots, \alpha_p, \Sigma)$ distribution if $b_{jl} = 0$ and $b_{jj} = \alpha_j$.
  - It is reduced to the $MCAR(\alpha, \Sigma)$ distribution if $b_{jl} = 0$ and $b_{jj} = \alpha$.
  - $MCAR(B, \Sigma)$ distribution is the most general of the three.

$MCAR(B, \Sigma)$ model

Hierarchical model: $Y_{ij} \overset{ind}{\sim} f(y_{ij} \mid \boldsymbol{\beta}, \phi_{ij}), \; i = 1, \ldots n \, , j = 1, \ldots p \, .$

$$\boldsymbol{\phi} \;\sim\; N_{np}\left(\mathbf{0}, \; (A \otimes I_{n\times n})\,(I_{p\times p} \otimes D - B \otimes W)^{-1}\,(A \otimes I_{n\times n})'\right)$$

- The distribution of ϕ is invariant to the choice of transformation matrix A (up to a reparametrization of B): let $T = AP$ such that $TT' = APP'A' = \Sigma$, and $C = P'BP$ (P is a $p \times p$ orthogonal matrix),

$$(T \otimes I)\,(I \otimes D - C \otimes W)^{-1}\,(T \otimes I)' = (A \otimes I)\,(I \otimes D - B \otimes W)^{-1}\,(A \otimes I)'$$

$MCAR(B,\, \Sigma)$ **model**

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- The conditions for having a valid joint distribution of ϕ:
  $\xi_i \zeta_j < 1,\;\; i.e.,\;\; \frac{1}{\xi_{min}} < \zeta_j < \frac{1}{\xi_{max}}$, where ξ_{min} and ξ_{max} are the minimum and maximum eigenvalues of $D^{-\frac{1}{2}} W D^{-\frac{1}{2}}$ (ξ_{max}=1 and $-1 < \xi_{min} < 0$), and ζ_j are the eigenvalues for B.

$$I_{p\times p}\otimes D - B\otimes W = (I_{p\times p}\otimes D)^{\frac{1}{2}} \left(I_{pn\times pn} - B \otimes D^{-\frac{1}{2}} W D^{-\frac{1}{2}}\right) (I_{p\times p}\otimes D)^{\frac{1}{2}}$$

Bayesian computing of $MCAR(B, \Sigma)$ model

Hierarchical model: $Y_{ij} \stackrel{ind}{\sim} f(y_{ij} \mid \boldsymbol{\beta}, \phi_{ij}),\ \ i = 1, \ldots n\,, j = 1, \ldots p\,.$

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- The eigenvalues ζ_j for B are in the range of $(\frac{1}{\xi_{min}},\ \frac{1}{\xi_{max}})$.

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- The eigenvalues ζ_j for B are in the range of $(\frac{1}{\xi_{min}},\; \frac{1}{\xi_{max}})$.
  - Spectral decomposition $B = P\Delta P'$, where P is the orthogonal matrix of eigenvectors and Δ is a diagonal matrix of eigenvalues ζ_j.
  - Parameterize the $p \times p$ orthogonal matrix P in terms of the $p(p-1)/2$ Givens angles θ_{ij} for $i = 1, \ldots, p-1$ and $j = i+1, \ldots, p$.
  - $P = G_{12}G_{13}\ldots G_{1p}\ldots G_{(p-1)p}$, G_{ij} is the $p \times p$ identity matrix with the ith and jth diagonal elements replaced by $\cos(\theta_{ij})$, and the (i, j) and (j, i) elements replaced by $\pm\sin(\theta_{ij})$, respectively.
  - Uniform$(-\pi/2, \pi/2)$ prior on the θ_{ij} and Uniform$(\frac{1}{\xi_{min}}, \frac{1}{\xi_{max}})$ prior on the ζ_j.

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  - Uniform$(-\pi/2, \pi/2)$ prior on the θ_{ij} and Uniform$(\frac{1}{\xi_{min}}, \frac{1}{\xi_{max}})$ prior on the ζ_j.
- $\Sigma^{-1} \sim Wishart\left(\nu, (\nu R)^{-1}\right)$.

  Then the prior distribution $p(A)$ becomes
  $p(A) \propto |AA'|^{-\frac{\nu+4}{2}} exp\left\{-\frac{1}{2}tr[\nu R(AA')^{-1}]\right\} \left|\frac{\partial\Sigma}{\partial a_{ij}}\right|$.

Simulation studies

Hierarchical model: $Y_{ij} \stackrel{ind}{\sim} N(\beta_j + \phi_{ij},\, \sigma^2)\,,\ i = 1, \ldots, 87,\ \ j = 1, 2.$

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- Study 1: $\phi \sim MCAR(B,\, \Sigma)$ with the true parameter values as follows: $a_{11} = 0.3$, $a_{12} = 0.1$, $a_{21} = 0$, $a_{22} = 0.3$, $b_{11} = 0.8$, $b_{12} = b_{21} = 0.4$ and $b_{22} = 0.1$.
- Study 2: $\phi \sim MCAR(\alpha,\, \Sigma)$ with the true parameter values as follows: $a_{11} = 0.3$, $a_{12} = 0.1$, $a_{21} = 0$, $a_{22} = 0.3$ and $\alpha = 0.8$.
- Study 3: $\phi \sim GMCAR(\alpha_1, \alpha_2, \eta_0, \eta_1, \tau_1, \tau_2)$ under the conditioning order $\boldsymbol{\phi}_1 \mid \boldsymbol{\phi}_2$, with the true parameter values as follows: $\alpha_1 = 0.1$, $\alpha_2 = 0.8$, $\eta_0 = 0.4$, $\eta_1 = 0.3$, $\tau_1 = 10$ and $\tau_2 = 10$.

Simulation results: AMSE

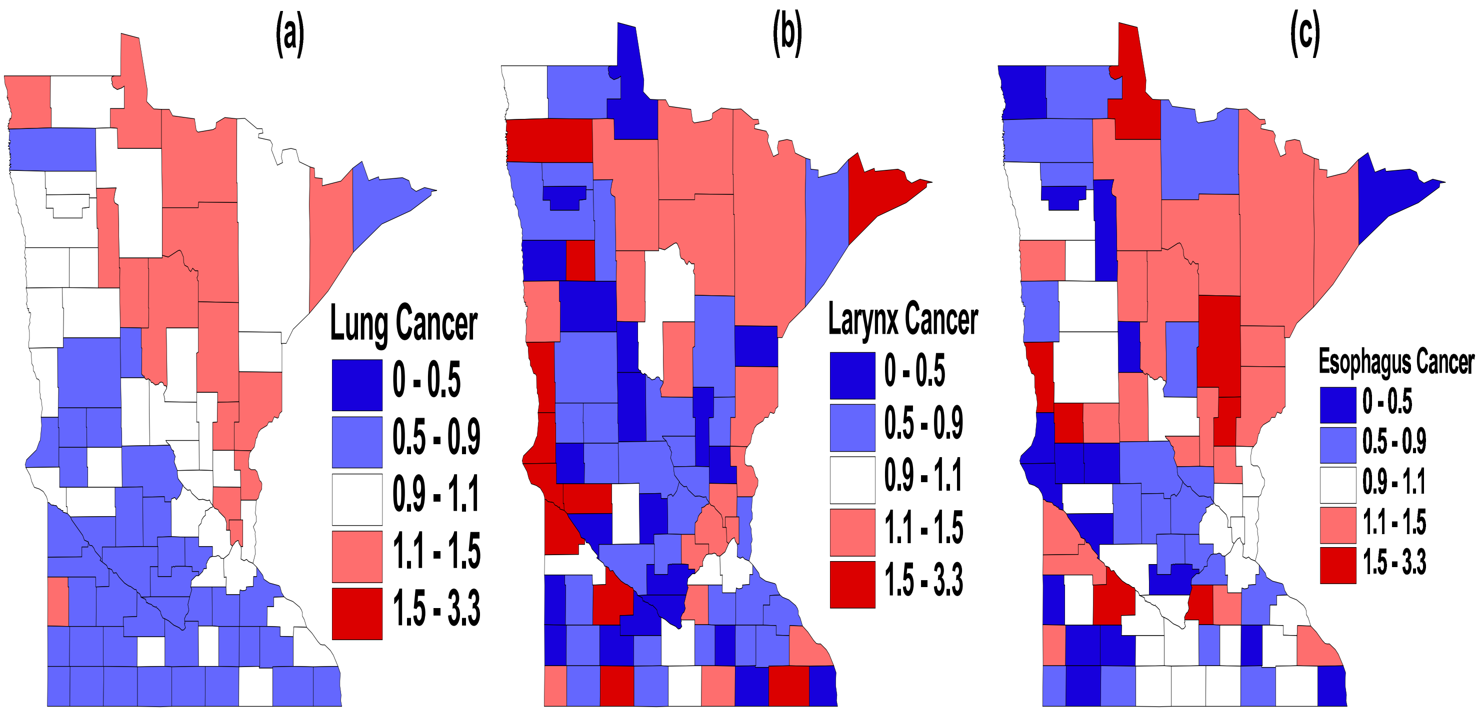
Average mean squared error (AMSE, $\times 10^{-3}$), associated Monte Carlo standard errors (se, $\times 10^{-5}$), and percentage change in AMSE ($\triangle, \%$) relative to the true model in each study based on N=1, 000 data sets. The symbol "–" indicates the true model for this study.

		$AMSE_1$		$AMSE_2$		overall $AMSE$	
	model	mean (se)	$\triangle$	mean (se)	$\triangle$	mean (se)	$\triangle$
1	1. $MCAR(B, \Sigma)$	7.06 (3.43)	–	6.81 (3.30)	–	6.93 (2.38)	–
	2. $MCAR(\alpha, \Sigma)$	7.20 (3.51)	1.98	7.38 (3.60)	8.37	7.29 (2.51)	5.29
	3. GMCAR	7.29 (3.55)	3.26	7.63 (3.73)	12.0	7.46 (2.58)	7.65
	4. GMCAR(reverse)	7.15 (3.48)	1.27	6.85 (3.31)	0.59	7.00 (2.40)	1.01
	5. bivariate I.I.D.	8.32 (4.12)	17.8	7.17 (3.59)	5.29	7.74 (2.74)	11.7
2	1. $MCAR(B, \Sigma)$	7.07 (3.44)	0.00	6.91 (3.33)	0.14	6.99 (2.40)	0.00
	2. $MCAR(\alpha, \Sigma)$	7.07 (3.45)	–	6.90 (3.33)	–	6.99 (2.40)	–
	3. GMCAR	7.20 (3.51)	1.84	7.28 (3.54)	5.51	7.24 (2.49)	3.62
	4. GMCAR(reverse)	7.37 (3.61)	4.24	7.02 (3.39)	1.74	7.19 (2.48)	2.90
	5. bivariate I.I.D.	8.28 (4.25)	17.1	8.04 (4.10)	16.5	8.16 (2.96)	17.0
3	1. $MCAR(B, \Sigma)$	7.66 (3.71)	2.00	6.70 (3.25)	–0.59	7.18 (2.47)	0.84
	2. $MCAR(\alpha, \Sigma)$	8.65 (4.24)	15.2	7.90 (3.87)	17.2	8.28 (2.87)	16.3
	3. GMCAR	7.51 (3.64)	–	6.74 (3.29)	–	7.12 (2.45)	–
	4. GMCAR(reverse)	7.78 (3.79)	3.60	6.79 (3.30)	0.74	7.28 (2.51)	2.30
	5. bivariate I.I.D.	8.61 (4.19)	14.6	7.48 (3.81)	11.0	8.04 (2.83)	12.9

Example: Minnesota cancer data

Maps of raw age-adjusted standardized mortality ratios (SMR) of lung, larynx and esophagus cancer in the years from 1990 to 2000 in Minnesota counties.

$E_{ij} = \sum_{k=1}^{m} \omega_j^k N_i^k$, $i = 1, \ldots, 87,\ j = 1, 2, 3$, $\omega_j^k = \sum_{i=1}^{87} D_{ij}^k / \sum_{i=1}^{87} N_i^k$ is the age-specific death rate due to cancer j for age group k over all counties.



Example: Minnesota cancer data (Cont'd)

Poisson regression model

$$Y_{ij} \stackrel{ind}{\sim} Poisson(E_{ij} e^{\beta_j + \phi_{ij}}) \; , \; i = 1, \ldots, 87, \; j = 1, 2, 3 \; ,$$

where Y_{ij} is the observed number of deaths due to cancer j in county i, and E_{ij} is the corresponding *age-adjusted* expected number of deaths (assumed known).

Example: Minnesota cancer data (Cont'd)

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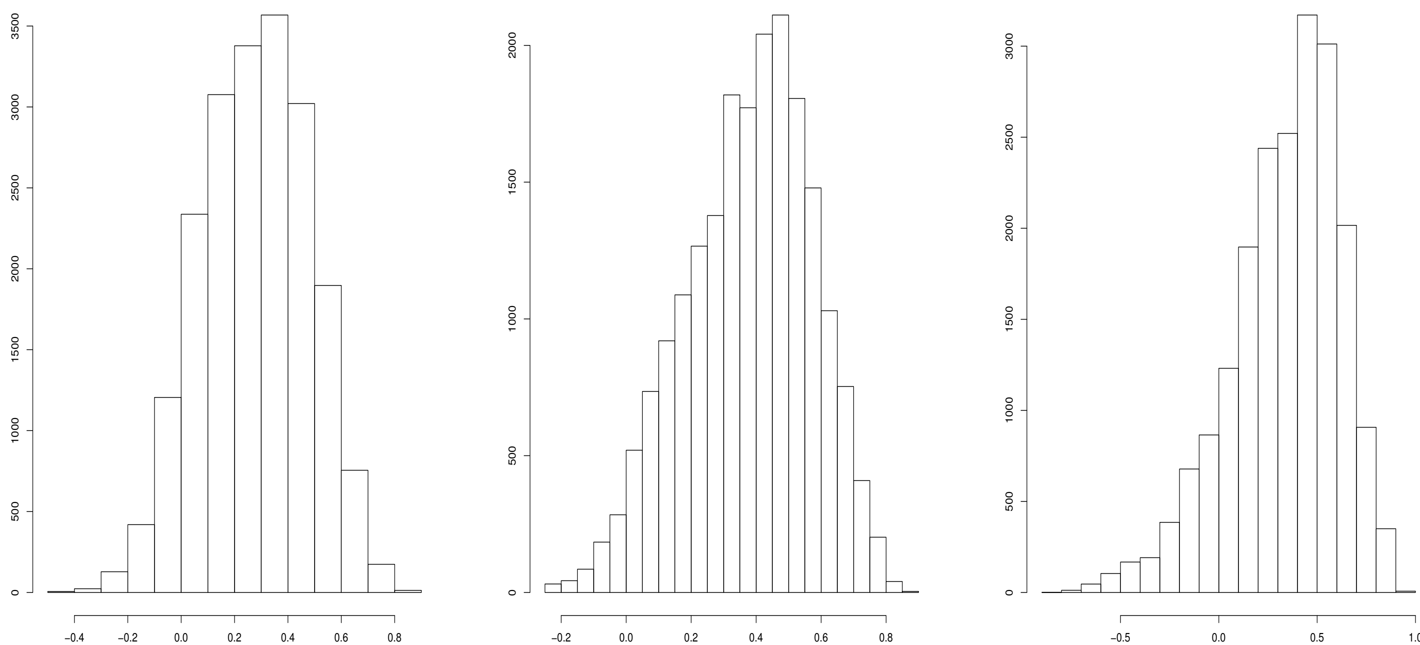
Model comparison by DIC statistics for Minnesota cancer data

	model	$\overline{D}$	p_D	DIC
1	$MCAR(B, \Sigma)$	138.8	82.5	221.3
2	$MCAR(\alpha_1, \alpha_2, \alpha_3, \Sigma)$	139.6	86.4	226.0
3	$MCAR(\alpha, \Sigma)$	143.4	81.9	225.3
4	$MCAR(\alpha = 1, \Sigma)$	147.6	80.5	228.1
5	separate CAR	147.6	82.8	230.4
6	trivariate I.I.D.	146.8	91.3	238.1

Example: Minnesota cancer data (Cont'd)

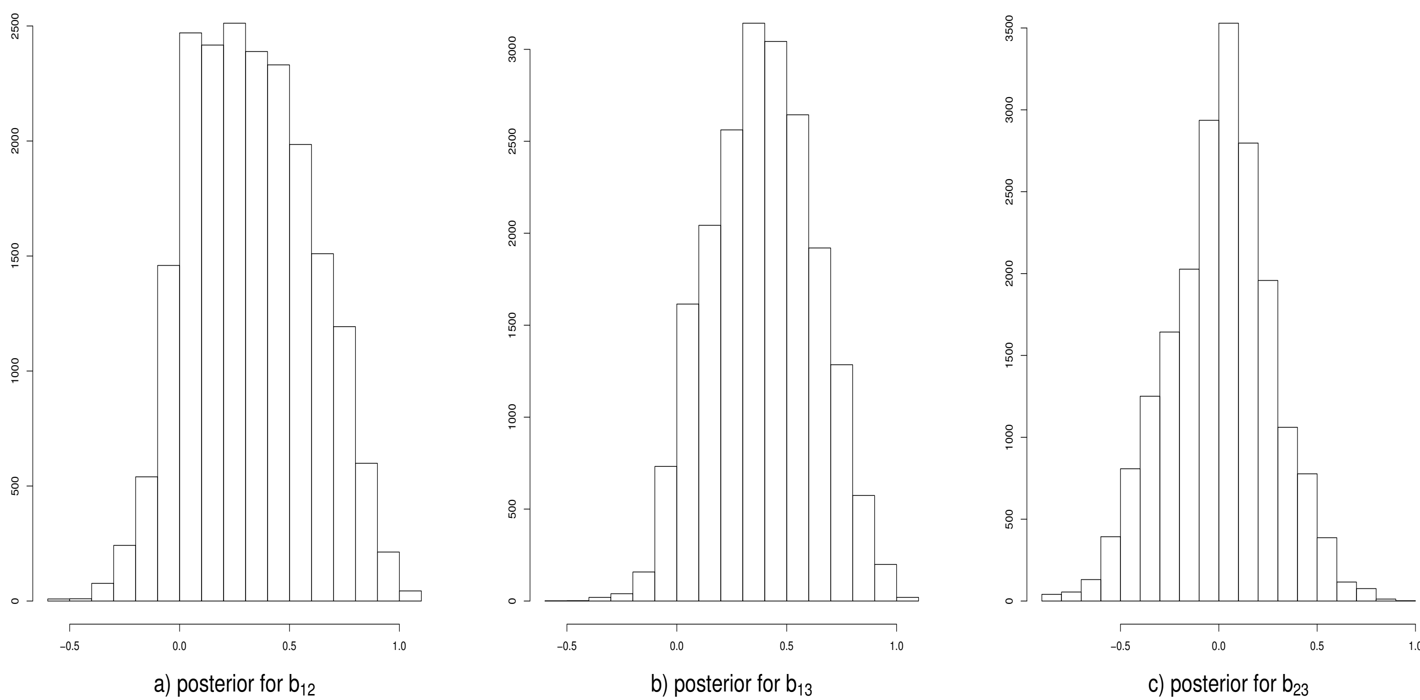
Posterior samples of ρ_{12}, ρ_{13} and ρ_{23} in the Minnesota cancer data analysis using the $MCAR(B,\,\Sigma)$ model: (a) estimated posterior for correlation ρ_{12} between lung and larynx; (b) estimated posterior for correlation ρ_{13} between lung and esophagus; (c) estimated posterior for correlation ρ_{23} between larynx and esophagus. $(\rho_{ij} = \Sigma_{ij}/\sqrt{\Sigma_{ii}\Sigma_{jj}})$



a) posterior for ρ_{12} b) posterior for ρ_{13} c) posterior for ρ_{23}

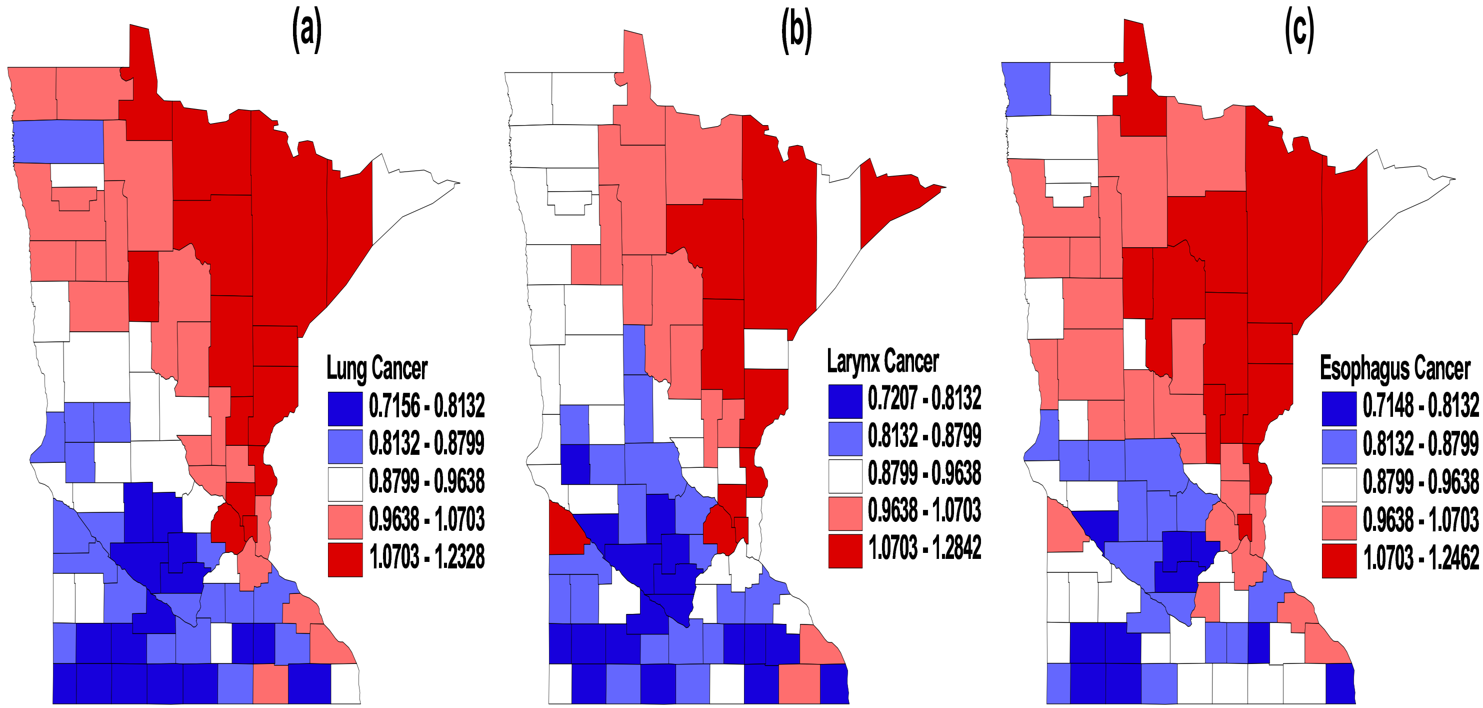
Example: Minnesota cancer data (Cont'd)

Posterior samples of b_{12}, b_{13} and b_{23} in the Minnesota cancer data analysis using the $MCAR(B,\ \Sigma)$ model: (a) estimated posterior for b_{12}; (b) estimated posterior for b_{13}; (c) estimated posterior for b_{23}.



Example: Minnesota cancer data (Cont'd)

Maps of posterior means of the fitted standard mortality ratios (SMR) of lung, larynx and esophagus cancer in the years from 1990 to 2000 in Minnesota from $MCAR(B, \Sigma)$ model:



Ongoing research

- Dynamic MCAR models for multivariate spatiotemporal data

$$Y_{ijt} \overset{ind}{\sim} Poisson(E_{ijt} e^{\mathbf{x}'_{ijt}\boldsymbol{\beta}_{jt} + \phi_{ijt}}) \;,\;\; i = 1, \ldots, n, \; j = 1, \ldots, p, \; t = 1, \ldots, T,$$

where $\boldsymbol{\phi}_{j,\, t+1} = H_j \boldsymbol{\phi}_{j,\, t} + \epsilon_{jt}$, $H_j = H = \theta_0 I + \theta_1 W$, and $\epsilon_t = (\epsilon'_{1t}, \ldots, \epsilon'_{pt})'$ follows a multivariate lattice model such as $MCAR(B_t,\, \Sigma_t)$ or $MCAR(B,\, \Sigma)$.

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  where $\boldsymbol{\phi}_{j,\,t+1} = H_j \boldsymbol{\phi}_{j,\,t} + \epsilon_{jt}$, $H_j = H = \theta_0 I + \theta_1 W$, and $\epsilon_t = (\epsilon'_{1t}, \ldots, \epsilon'_{pt})'$ follows a multivariate lattice model such as $MCAR(B_t,\, \Sigma_t)$ or $MCAR(B,\, \Sigma)$.

- Spatially varying coefficients model

$$Y_i \overset{ind}{\sim} Poisson(E_i e^{\mathbf{x}'_i\boldsymbol{\beta} + \varsigma_{i1} z_{i1} + \varsigma_{i2} z_{i2} + \phi_i}), \; i = 1, \ldots, n,$$

  where $\varsigma_{\mathbf{1}} = (\varsigma_{11}, \ldots, \varsigma_{n1})'$, $\varsigma_{\mathbf{2}} = (\varsigma_{12}, \ldots, \varsigma_{n2})'$, $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)'$. $\Upsilon = (\varsigma_{\mathbf{1}}', \varsigma_{\mathbf{2}}', \boldsymbol{\phi}')'$ follows a multivariate lattice model such as $MCAR(B,\, \Sigma)$.

Ongoing research

- Dynamic MCAR models for multivariate spatiotemporal data

$$Y_{ijt} \overset{ind}{\sim} Poisson(E_{ijt} e^{\mathbf{x}'_{ijt}\boldsymbol{\beta}_{jt} + \phi_{ijt}}) \ , \ \ i = 1, \ldots, n, \ j = 1, \ldots, p, \ t = 1, \ldots, T,$$

  where $\boldsymbol{\phi}_{j,\, t+1} = H_j \boldsymbol{\phi}_{j,\, t} + \epsilon_{jt}$, $H_j = H = \theta_0 I + \theta_1 W$, and $\epsilon_t = (\epsilon'_{1t}, \ldots, \epsilon'_{pt})'$ follows a multivariate lattice model such as $MCAR(B_t,\, \Sigma_t)$ or $MCAR(B,\, \Sigma)$.

- Spatially varying coefficients model

$$Y_i \overset{ind}{\sim} Poisson(E_i e^{\mathbf{x}'_i\boldsymbol{\beta} + \varsigma_{i1} z_{i1} + \varsigma_{i2} z_{i2} + \phi_i}) \ , \ \ i = 1, \ldots, n,$$

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- Spatial factor analysis with p factors $\mathbf{u}_j = (v_{1j}, \ldots, v_{nj})'$, $j = 1, \ldots, p$

$$Y_{ij} \sim f(y_{ij} \mid \boldsymbol{\beta}, \phi_{ij}), \ \ i = 1, \ldots n \,, j = 1, \ldots m \ \ (m > p) \,,$$

  where $\boldsymbol{\phi} = (A \otimes I_{n \times n})\mathbf{u}$, $\mathbf{u} = (\mathbf{u}'_1, \ldots, \mathbf{u}'_p)'$, and A is a $m \times p$ loading matrix.

Ongoing research

- Dynamic MCAR models for multivariate spatiotemporal data

$$Y_{ijt} \stackrel{ind}{\sim} Poisson(E_{ijt}e^{\mathbf{x}'_{ijt}\boldsymbol{\beta}_{jt} + \phi_{ijt}}) \ , \ \ i = 1, \ldots, n, \ j = 1, \ldots, p, \ t = 1, \ldots, T,$$

  where $\boldsymbol{\phi}_{j,\ t+1} = H_j \boldsymbol{\phi}_{j,\ t} + \epsilon_{jt}$, $H_j = H = \theta_0 I + \theta_1 W$, and $\epsilon_t = (\epsilon'_{1t}, \ldots, \epsilon'_{pt})'$ follows a multivariate lattice model such as $MCAR(B_t,\ \Sigma_t)$ or $MCAR(B,\ \Sigma)$.

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$$Y_i \stackrel{ind}{\sim} Poisson(E_i e^{\mathbf{x}'_i\boldsymbol{\beta} + \varsigma_{i1} z_{i1} + \varsigma_{i2} z_{i2} + \phi_i}) \ , \ \ i = 1, \ldots, n,$$

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- Some other applications of multivariate lattice models

Acknowledgements

Thank you very much for your attention!

- The papers about this talk can be found in *Research Report* rr2003-002, rr2004-001 and rr2005-001, Division of Biostatistics, University of Minnesota.
- Software related to some multivariate lattice models can be found from:
  `http://www.biostat.umn.edu/~sudiptob`.
  `http://www.biostat.umn.edu/~brad`.
- Email me at sudiptob@biostat.umn.edu for any further question.