Spatio-temporal Models

- Again point-referenced vs. areal unit data
- Continuous time vs. discretized time

$\Rightarrow$ association in space, association in time
For point-referenced data, t continuous, Gaussian
$Y(\mathbf{s},t) = \mu(\mathbf{s},t) + w(\mathbf{s},t) + \epsilon(\mathbf{s},t)$

non-Gaussian data, $g(EY(\mathbf{s},t) = \mu(\mathbf{s},t) + w(\mathbf{s},t)$

Don't treat time as a third coordinate $(\mathbf{s},t)$

$$Cov(Y(\mathbf{s},t), Y(\mathbf{s}',t')) = C(\mathbf{s}-\mathbf{s}', t-t')$$

Spatio-temporal Models

- Separable form:

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- Nonseparable form:
  - Sum of independent separable processes
  - Mixing of separable covariance functions
  - Spectral domain approaches

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  - E, residuals matrix after a regression fitting, Empirical orthogonal functions (EOF)

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- Modeling: $Y_t(\mathbf{s}) = \mu_t(\mathbf{s}) + w_t(\mathbf{s}) + \epsilon_t(\mathbf{s})$,
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- For $\epsilon_t(\mathbf{s})$, i.i.d. $N(0, \tau_t^2)$
- For $w_t(\mathbf{s})$
  - $w_t(\mathbf{s}) = \alpha_t + w(\mathbf{s})$
  - $w_t(\mathbf{s})$ independent for each t
  - $w_t(\mathbf{s}) = w_{t-1}(\mathbf{s}) + \eta_t(\mathbf{s})$, independent spatial process innovations

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- $Y_i(t)$, temporal process for each unit (rare!)
  Y_{it}, more common, a time series for each unit
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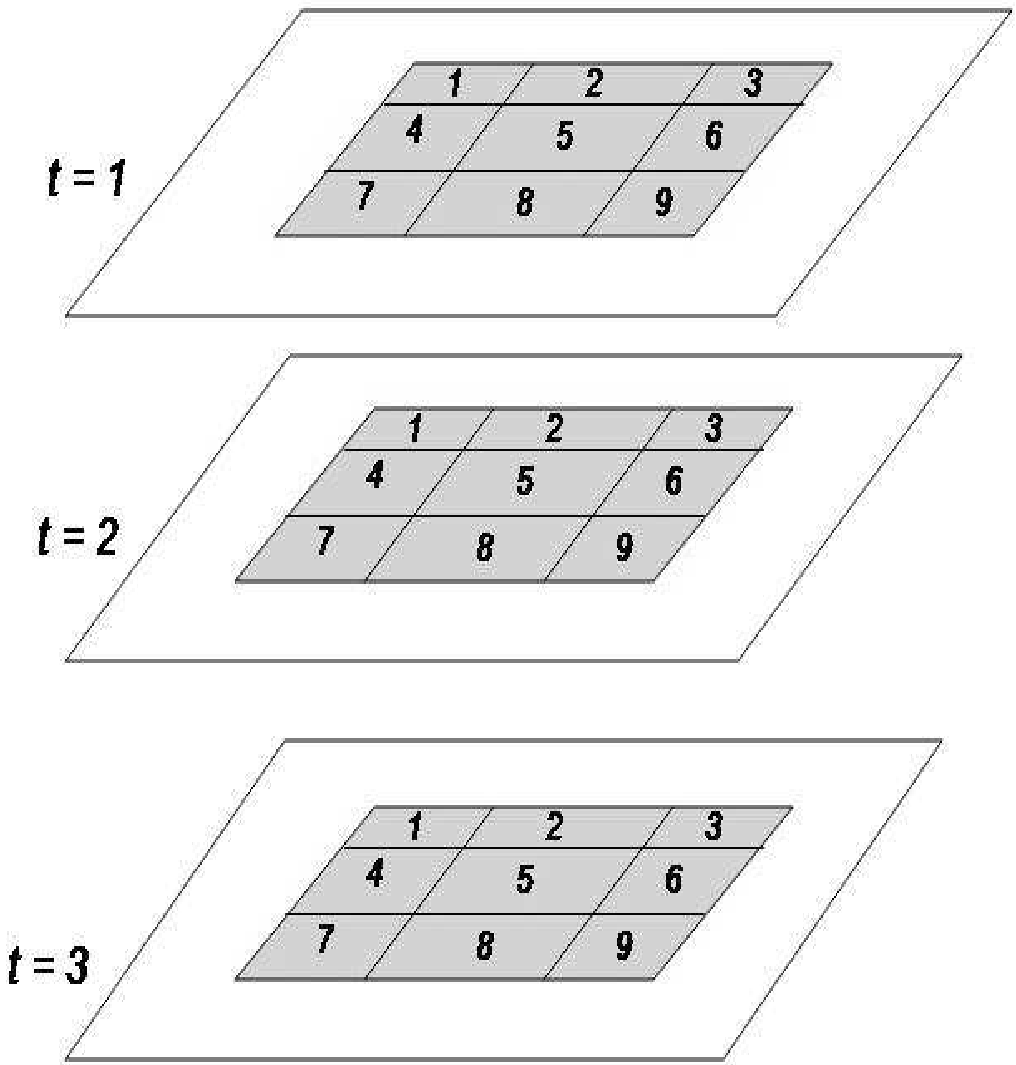
- Modeling for ϕ_{it}?? CAR in space and time
  - Space nested within time $\phi_i^{(t)}$, CAR τ_t^2 (λ_t, indep across t)
  - $\phi_{it}|\boldsymbol{\phi}_{-(it)}$, space, time neighbors, weight for space, weight for time
  - MCAR, $\boldsymbol{\phi}_i = (\phi_{i1}, \phi_{i2}, ...\phi_{iT})$, short series

Neighbors in time and space



Spatiotemporal Dynamic Models

- Consider a collection of sites $S = \{\mathbf{s}_1, ..., \mathbf{s}_{N_s}\}$, and time-points $T = \{t_1, ..., t_{N_t}\}$, yielding observations $Y(\mathbf{s}, t)$, and p regressors, $\mathbf{x}(\mathbf{s}, t)$, for every $(\mathbf{s}, t) \in S \times T$.

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- Dynamic Equation:

  Measurement Equation:

$$Y(\mathbf{s}, t) = \mu(\mathbf{s}, t) + \epsilon(\mathbf{s}, t); \ \epsilon(\mathbf{s}, t) \overset{ind}{\sim} N\left(0, \sigma_\epsilon^2\right)$$

$$\mu(\mathbf{s}, t) = \mathbf{x}^T(\mathbf{s}, t)\,\tilde{\boldsymbol{\beta}}(\mathbf{s}, t).$$

$$\tilde{\boldsymbol{\beta}}(\mathbf{s}, t) = \boldsymbol{\beta}_t + \boldsymbol{\beta}(\mathbf{s}, t)$$

  Transition Equation:

$$\boldsymbol{\beta}_t = \boldsymbol{\beta}_{t-1} + \boldsymbol{\eta}_t; \ \boldsymbol{\eta}_t \overset{ind}{\sim} N_p\left(\mathbf{0}, \Sigma_{\boldsymbol{\eta}}\right)$$

$$\boldsymbol{\beta}(\mathbf{s}, t) = \boldsymbol{\beta}(\mathbf{s}, t-1) + \mathbf{w}(\mathbf{s}, t).$$

Spatiotemporal Processes

- How do we model the process $\mathbf{w}(\mathbf{s},t)$? Note that $\mathbf{w}(\mathbf{s},t)$ is a multivariate process – it has p component processes, one for each regression coefficient. So, we set $\mathbf{w}(\mathbf{s},t) \sim MVGP(\mathbf{0}, C(\mathbf{s},\mathbf{s}';t,t'))$, where:

$$C(\mathbf{s},\mathbf{s}';t,t') = [Cov(w_i(\mathbf{s},t), w_j(\mathbf{s}',t'))]_{i,j=1}^{p}.$$

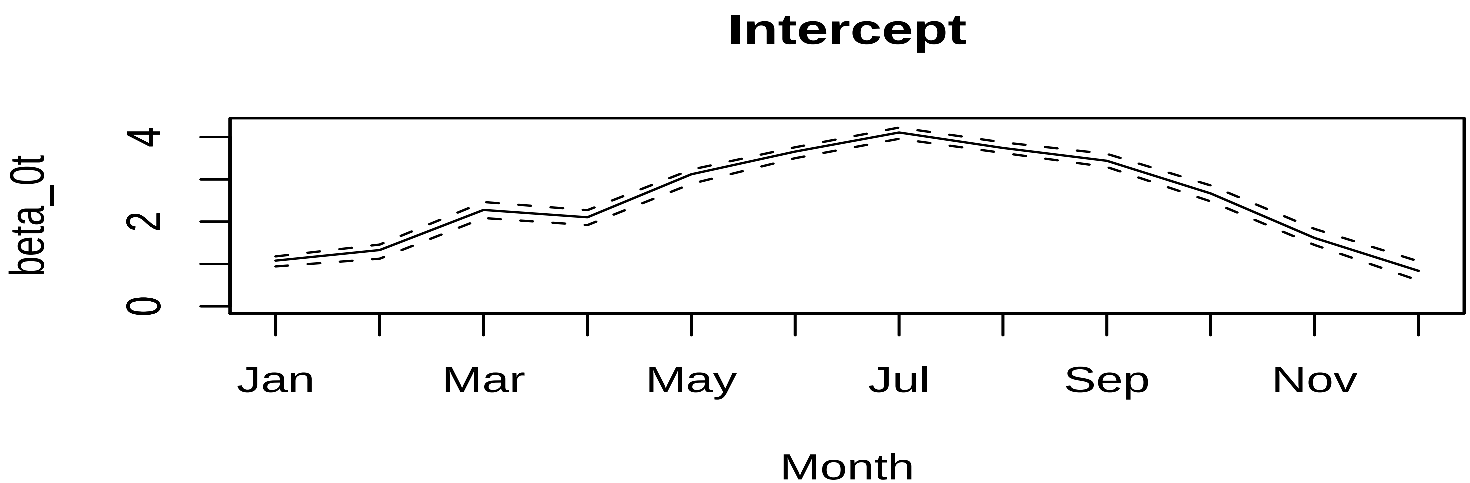
- Simpler *separable* modeling:

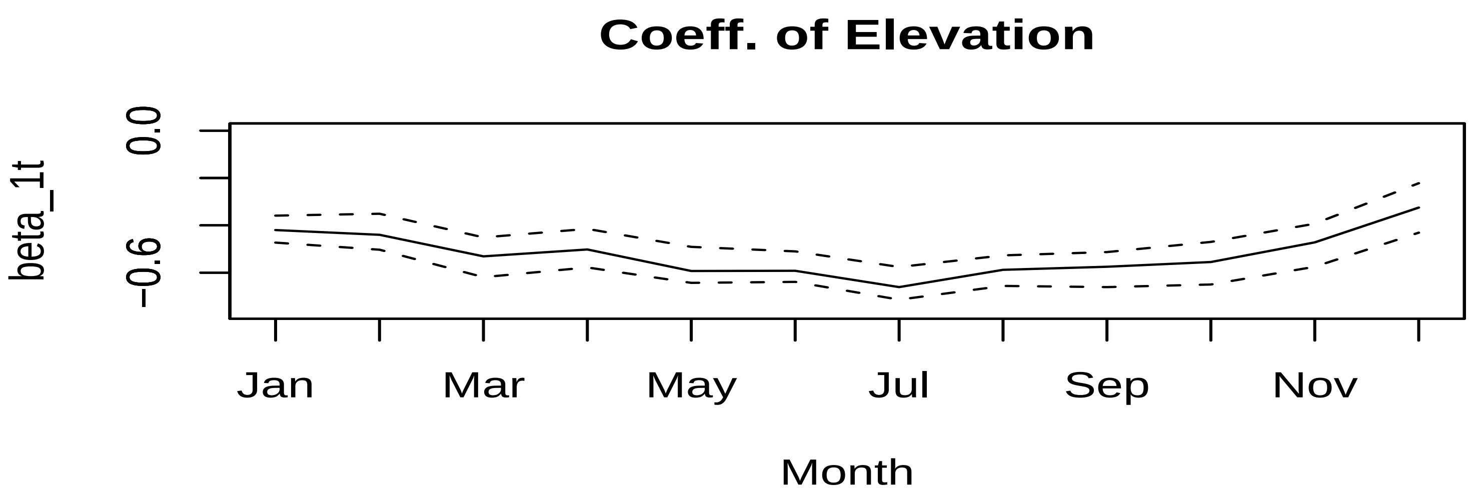
$$C(\mathbf{s},\mathbf{s}';t,t') = \rho_{space}(\mathbf{s},\mathbf{s}';\phi_{space})\rho_{time}(t,t';\phi_{time})\Lambda$$

  where Λ is a common $p \times p$ covariance matrix for the components of $\mathbf{w}(\mathbf{s},t)$ *within* each site-time combination.

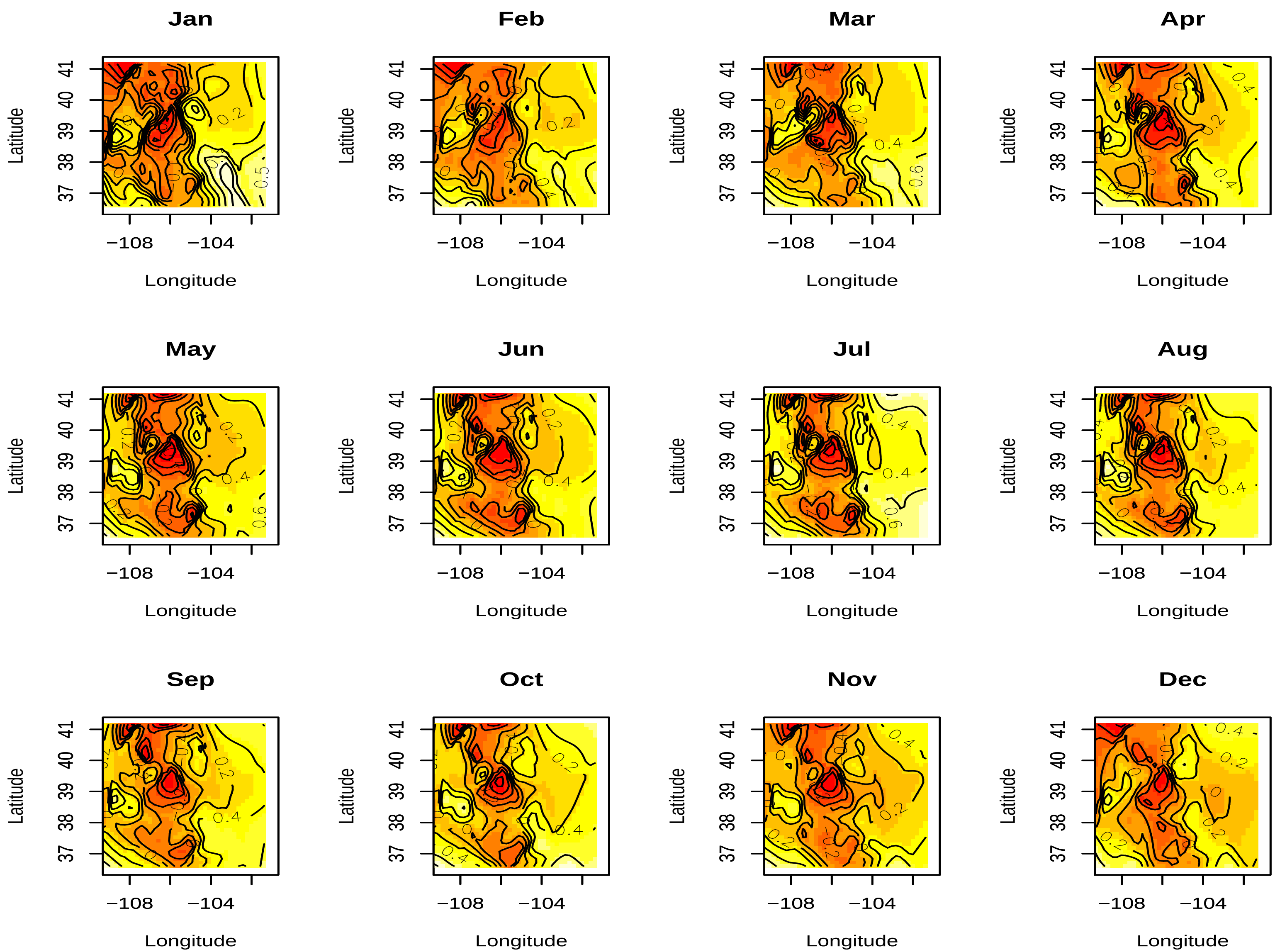
- One approach: $\mathbf{w}(\mathbf{s},t) = A\mathbf{v}(\mathbf{s},t)$ where $\mathbf{v}(\mathbf{s},t) \sim MVGP(\mathbf{0}, \oplus_{l=1}^{p}(\rho_l(\mathbf{s},\mathbf{s}';\phi_l)))$.

Temperature given Elevation

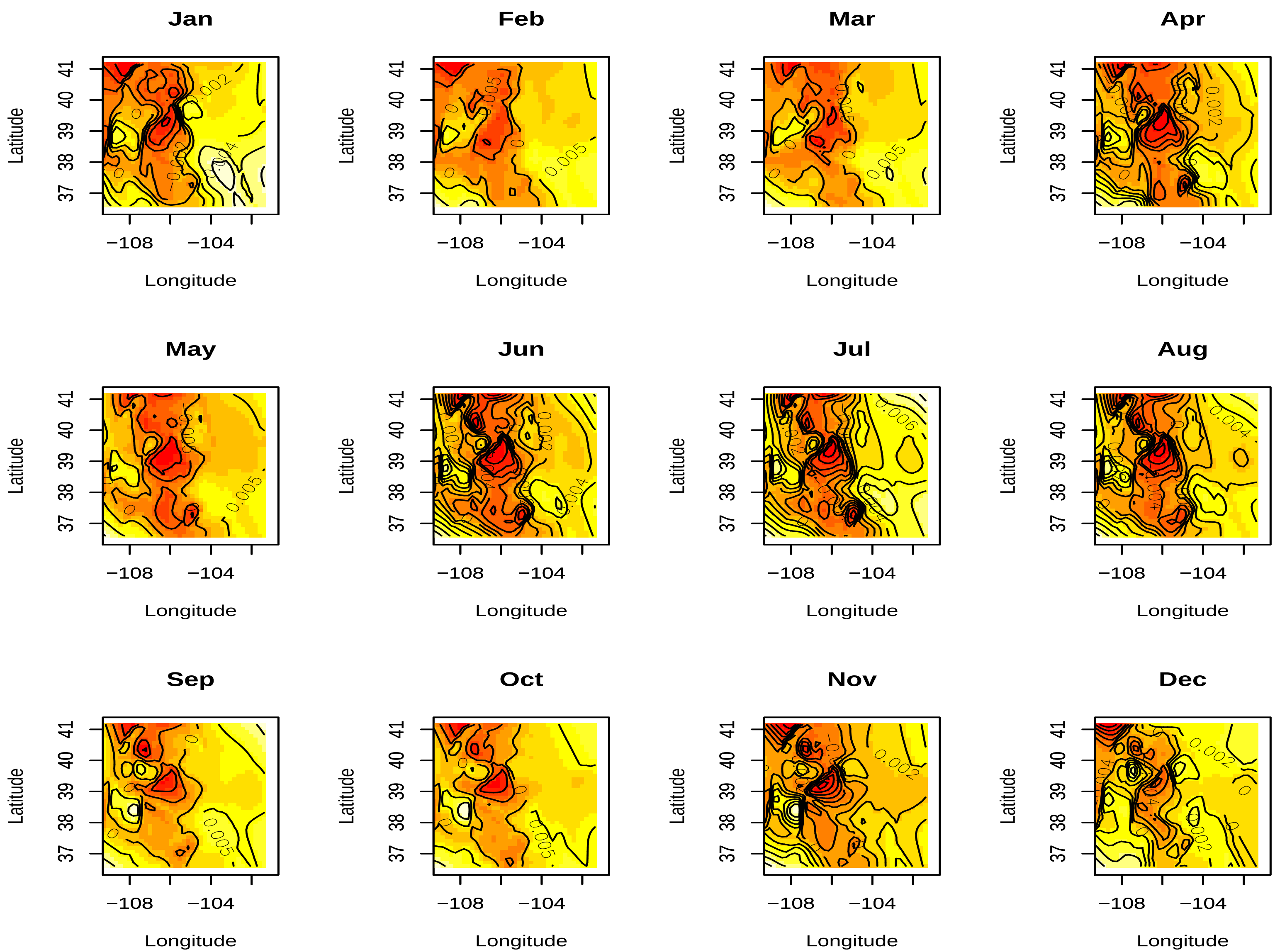




Spatially varying Intercept Process



Spatially varying Slope Process



Predictive Performance