Poisson Process HW

1. Let $X(t)$ be a Poisson process with parameter λ. Independently, let T be a random variable with the exponential density

$$f_T(t) = \theta \exp(-\theta t) 1(t > 0).$$

Determine the probability mass function of $X(T)$.

2. Customers arrive at a holding facility at random according to a Poisson process having rate λ. The facility processes in batches of size Q. That is, the first $Q-1$ customers wait until the arrival of the Q^{th} customer. Then all are passed simultaneously, and the process repeats. Service times are instantaneous. Let $N(t)$ be the number of customers in the holding facility at time t. Assume that $N(0) = 0$ and let $T = \min\{t \geq 0 : N(t) = Q\}$ be the first dispatch time. Find $E[T]$ and $E\left[\int_0^T N(t)\,dt\right]$.

3. Suppose a device is exposed to one of k possible environments $E_1, \ldots, E_k$, which can occur with respective probabilities $\pi_1, \ldots, \pi_k$, $\left(\sum_{i=1}^k \pi_i = 1\right)$. In each environment dangerous peaks occur according to a Poisson process with parameter λ_j, $j = 1, \ldots, k$. Within the environment E_j, the conditional probability that the device fails, given that a peak occurs is p_j. Find the probability that a device fails within a given length of time t.

4. Let $S_1, S_2, \ldots, S_k$ be the event times for $X(t) \sim PP(\lambda)$. Suppose it is known that $X(1) = n$. For $k < n$, find the conditional distribution of $S_1, \ldots, S_{k-1}, S_{k+1}, \ldots, S_n$, given that $S_k = s$.

5. Let $S_1, S_2 \ldots$ be the event times for $X(t) \sim PP(\lambda)$. Independent of the process, let $Y_1, Y_2, \ldots$ be iid random variables with common pdf $f(x)$. Determine the distribution of

$$Z = \min\{S_1 + Y_1, \ldots, S_n + Y_n\}.$$

6. Points are placed on the surface of a circular disk of radius 1 according to the following scheme. First, a Poisson distributed random variable N is observed. If $N = n$, then n random variables $\theta_1, \ldots, \theta_n$ are independently generated, each $U(0, 2\pi)$, and n random variables $R_1, \ldots, R_n$ are independently generated with the triangular density,

$$f(r) = 2r1(0 < r < 1).$$

Finally the points are located at the positions with polar coordinates (R_i, θ_i), $i = 1, \ldots, n$. What is the distribution of the resulting point process on the disk?

7. Let $\left\{N(A); A \in R^2\right\}$ be a homogeneous $PP(\lambda)$. Suppose there is a reaction between two or more points whenever they are located within a distance d of one another. Determine the distribution of the number of reactions in a circle of radius r, valid in the limit as $r \to \infty$ and $d \to 0$ in such a way that $r^2 d \to \mu > 0$.

8. Consider spheres in 3-dimensional space with centers distributed according to a Poisson distribution with parameter $\lambda |A|$ where $|A|$ now represents the volume of the set A. If the radii of all spheres are distributed according to $F(r)$ with density $f(r)$ (with finite third moment), what do you think is the distribution of the random variable representing the number of spheres that cover a fixed point $\mathbf{x}$ in space? Characterize this distribution in terms of f, r and λ ($\mathbf{x}$ is not involved.)

9. Shocks occur to a system according to $PP(\lambda)$. Each shock causes some damage to the system, and these damages accumulate. Let $N(t)$ be the number of shocks up to time t, and let Y_i be the damage caused by the i^{th} shock. Then,

$$Z(t) = \sum_{i=1}^{N(t)} Y_i$$

is the total damage up to time t. Suppose that the system continues to operate as long as $Z(t) < a$, where a is some specified critical value, and fails when $Z(t) \geq a$. Find the mean time to system failure when the individual damages Y_i are iid having a geometric distribution with

$$P[Y = k] = \alpha (1 - \alpha)^{k-1}, k = 1, 2, \ldots$$

10. Let $X(t)$ and $Y(t)$ be independent Poisson processes with respective intensities λ and μ. For a fixed integer a, let $T_a = \min\{t \geq 0 : Y(t) = a\}$ be the random time the Y process first reaches the value a. Determine $P\{X(T_a) = k\}$ for $k = 0, 1, \ldots$.