Algebraic Models for Multilinear Dependence

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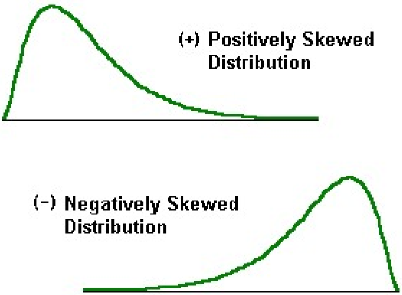
IMSNRC

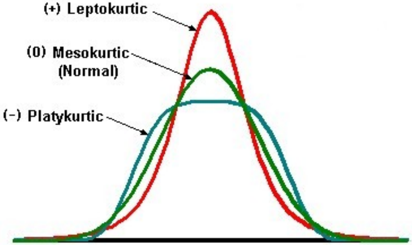
Joint work with Lek-Heng Lim of U.C. Berkeley

1. Introduction
2. Definitions
3. Properties
4. Principal Cumulant Component Analysis
5. Applications

Univariate cumulants

Mean, variance, skewness and kurtosis describe the shape of a univariate distribution.





Covariance matrices

The covariance matrix partly describes the dependence structure of a multivariate distribution.

- Principal Component Analysis
- Factor models
- Gaussian graphical models
- Risk–bilinear form computes variance $h^\top \Sigma h$ of holdings

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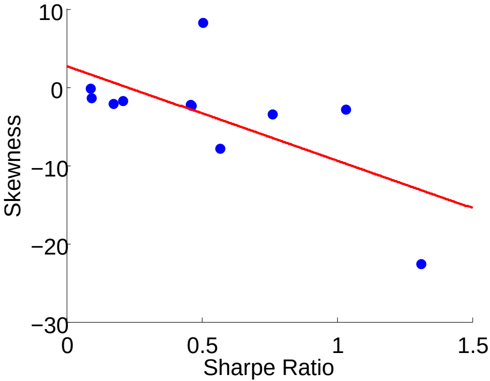
But if the variables are not multivariate Gaussian, not the whole story.

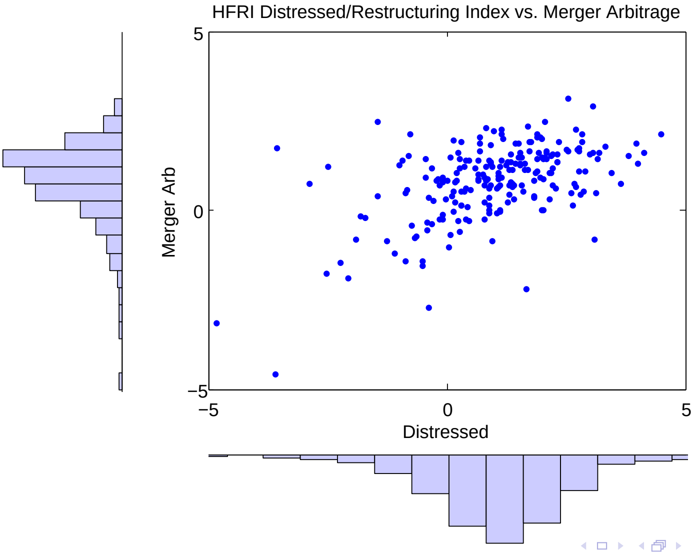
- In the financial markets, too much reliance on a quadratic, Gaussian perspective on risk.
- Exploited by trading skewness and kurtosis risk for apparent reduction in variance.

Sharpe Ratio ($\frac{\mu-\mu_f}{\sigma}$) vs Skewness



Hedge Fund Research Indices daily returns

Non-multivariate Gaussian returns are common;



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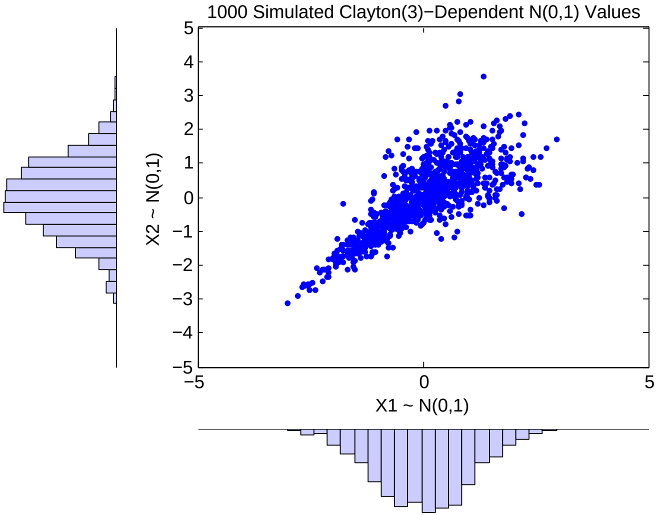

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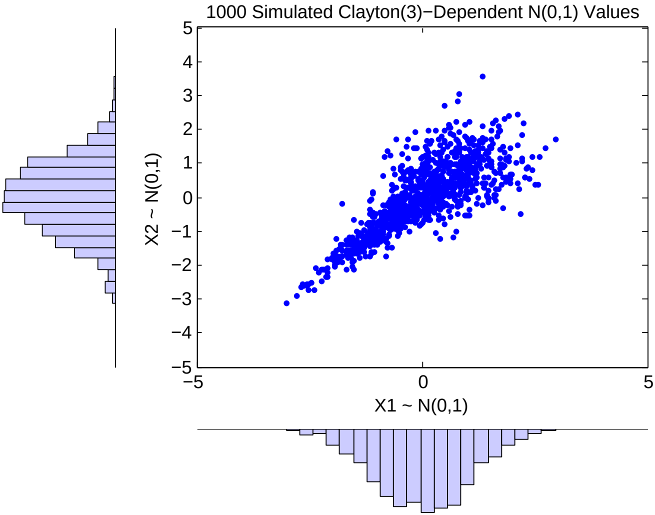

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6. Dimension reduction

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Symmetric tensors and actions

A tensor in coordinates is a multi-way array with a multilinear action.

- Tensor $[\![a_{ijk}]\!] \in \mathbb{R}^{r \times r \times r}$ is symmetric if it is invariant under all permutations of indices

$$a_{ijk} = a_{ikj} = a_{jik} = a_{jki} = a_{kij} = a_{kji}.$$

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Comes with an action:

- If C is $r\times r$, Q is $p\times r$, we have $K := Q \cdot C = QCQ^\top$

$$\kappa_{\ell,m} = \sum_{i,j=(1,1)}^{(r,r)} q_{\ell i} q_{mj} c_{ij}.$$

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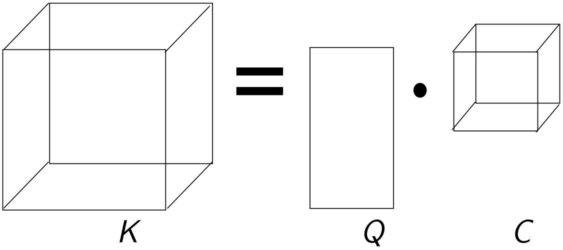
- For $d > 2$ multiply on $3, 4, \ldots$ "sides" of the multi-way array: *Symmetric multilinear matrix multiplication*.

Symmetric multilinear matrix multiplication



If Q is a $p \times r$ matrix, C an $r \times r \times r$ tensor, make a $p \times p \times p$ tensor $K = (Q, Q, Q) \cdot C$ or

$$K = Q \cdot C$$

$$\kappa_{\ell mn} = \sum_{i,j,k=(1,1,1)}^{(r,r,r)} q_{\ell i} q_{mj} q_{nk} c_{ijk}.$$

Moments and Cumulants are symmetric tensors

Vector-valued random variable $\mathbf{x} = (X_1, \ldots, X_p)$.
Three natural d-way tensors are:

- The dth non-central moment $s_{i_1,\ldots,i_d}$ of $\mathbf{x}$:

$$S_d(\mathbf{x}) = \left[\mathbb{E}(x_{i_1} x_{i_2} \cdots x_{id})\right]_{i_1,\ldots,i_d=1}^{p}.$$

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$$K_d(\mathbf{x}) = \left[\sum_{A_1 \sqcup \cdots \sqcup A_q = \{i_1,\ldots,i_d\}} (-1)^{q-1}(q-1)! s_{A_1} \ldots s_{A_q}\right]_{i_1,\ldots,i_d=1}^{p}.$$

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$s_{i_1,\ldots,i_d} = \sum_B \prod_{b \in B} \kappa_b$ and $\kappa_{ijk\ell} = m_{ijk\ell} - (m_{ij}m_{k\ell} + m_{ik}m_{j\ell} + m_{i\ell}m_{jk})$

Measuring useful properties.

For univariate x, the cumulants $K_d(x)$ for $d = 1, 2, 3, 4$ are

- expectation $\kappa_i = \mathbb{E}[x]$,
- variance $\kappa_{ii} = \sigma^2$,
- skewness $\kappa_{iii}/\kappa_{ii}^{3/2}$, and
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Alternative definitions in terms of log characteristic function, coefficients in multivariate Edgeworth series. See [Fisher 1929, McCullagh 1984,1987] for definitions and properties.

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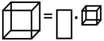
Properties of cumulants: Multilinearity

- Multilinearity: if $\mathbf{x}$ is a $\mathbb{R}^r$-valued random variable and $A \in \mathbb{R}^{p \times r}$

$$K_d(A\mathbf{x}) = A \cdot K_d(\mathbf{x}),$$

where $\cdot$ is the multilinear action  .

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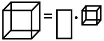
where $\cdot$ is the multilinear action  .

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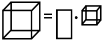
where $\cdot$ is the multilinear action  .

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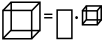
where $\cdot$ is the multilinear action  .

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- Independent Component Analysis [Comon 94] finds an A to approximately diagonalize $K_d(\mathbf{x})$.

Properties of cumulants: Independence

Independence:

- If $\mathbf{x}_1, \ldots, \mathbf{x}_p$ are random variables mutually independent of $\mathbf{y}_1, \ldots, \mathbf{y}_p$, we have
  $K_d(\mathbf{x}_1 + \mathbf{y}_1, \ldots, \mathbf{x}_p + \mathbf{y}_p) = K_d(\mathbf{x}_1, \ldots, \mathbf{x}_p) + K_d(\mathbf{y}_1, \ldots, \mathbf{y}_p)$.

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- $K_{i_1,\ldots,i_d}(\mathbf{x}) = 0$ whenever there is a partition of $\{i_1, \ldots, i_d\}$ into two nonempty sets I and J such that $\mathbf{x}_I$ and $\mathbf{x}_J$ are independent.

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- Why we want to diagonalize in independent component analysis

Properties of cumulants: Vanishing and Extending

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  - Parametrization is trickier when K_2 doesn't tell the whole story.

Making cumulants useful, tractable and estimable

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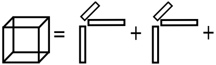
PCA: eigenvalue decomposition of a positive semidefinite real symmetric matrix. We need a tensor analog.

But, it isn't as easy as it looks...

Tensor decomposition

Three possible generalizations are the same in the matrix case but not in the tensor case. For a $p \times p \times p$ tensor K,

Name	minimum r such that	
Tensor rank	$K = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$ not closed	

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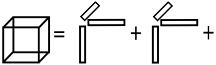
Tensor rank	$K = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$  not closed
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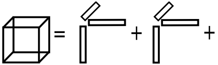
Tensor rank	$K = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$  not closed
Border rank	$K = \lim_{\epsilon \to 0}(S_\epsilon)$, Tensor rank$(S_\epsilon) = r$ closed but hard to represent; defining equations unknown.
Multilinear rank	$K = A \cdot C, C \in \mathbb{R}^{r \times r \times r}, A \in \mathbb{R}^{p \times r}$, closed and understood: subspace varieties.

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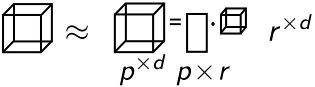
Multilinear rank factor model

Let $\mathbf{y} = Y_1, \ldots, Y_p$ be a random vector. Write the dth order cumulant $K_d(\mathbf{y})$ as a best r-multilinear rank approximation in terms of the cumulant $K_d(\mathbf{x})$ of a smaller set of r factors $\mathbf{x}$:

$$K_d(\mathbf{y}) \approx Q \cdot K_d(\mathbf{x})$$



where

- Q is orthonormal, and $Q^\top$ projects to the factors
- The column space of Q defines the r-dim subspace which best explains the dth order dependence.
- In place of eigenvalues, we have the core tensor $K_d(\mathbf{x})$, the cumulant of the factors, analogous to the covariance matrix of the factors in the $r \times r$ case.

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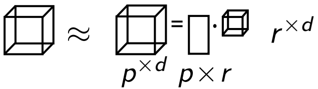

where

- Q is orthonormal, and $Q^\top$ projects to the factors
- The column space of Q defines the r-dim subspace which best explains the dth order dependence.
- In place of eigenvalues, we have the core tensor $K_d(\mathbf{x})$, the cumulant of the factors, analogous to the covariance matrix of the factors in the $r \times r$ case.

Have model, need loss and algorithm.

Principal cumulant component analysis 1

Factors/principal components that account for variation in each cumulant separately

$$\min_{Q \in \mathrm{O}(p,r),\ C_d \in \mathrm{S}^d(\mathbb{R}^r)} \|\hat{K}_d(\mathbf{y}) - Q \cdot C_d\|^2,$$

Minimize over

- $C_d \approx \hat{K}_d(\mathbf{x})$ a NOT-necessarily-diagonal small $(r \times r \times r)$ symmetric tensor.
- Q an orthonormal matrix

Principal cumulant component analysis 2

Or, factors/principal components that account for variation in all cumulants simultaneously

$$\min_{Q \in \mathrm{O}(p,r),\ C_d \in \mathrm{S}^d(\mathbb{R}^r)} \sum_{d=2}^{\infty} \alpha_d \|\hat{K}_d(\mathbf{y}) - Q \cdot C_d\|^2,$$

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- Reduces to optimization over a single Grassmannian (set of r-dim spaces in p-dim space) of dimension $r(p-r)$,

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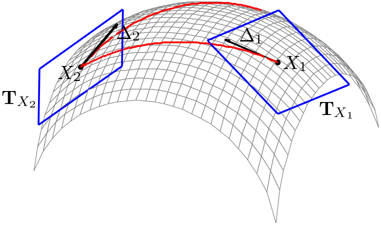
- In practice $\infty = 3$ or 4.

Estimation

[Arias, Edelman, Smith; 1999] showed how to reduce gradient-based optimization (parallel transport, etc.) on Grassmann manifold to numerical linear algebra.



Quasi-Newton, L-BFGS on the Grassmannian are effective for tensor problems [Eldén, Savas; 2008], [Savas, Lim; 2008].

1 Introduction

2 Definitions

3 Properties

4 Principal Cumulant Component Analysis

5 Applications

Mean-variance portfolio optimization

Markowitz mean-variance portfolio optimization defines risk to be variance. For portfolio holdings h, solve

$$\min h^\top K_2(\mathbf{x}) h \qquad s.t. \qquad h^\top \mathbb{E}[\mathbf{x}] > \underline{r}$$

Evidence indicates that investors optimizing variance with respect to the covariance matrix accept unwanted skewness and kurtosis risk.

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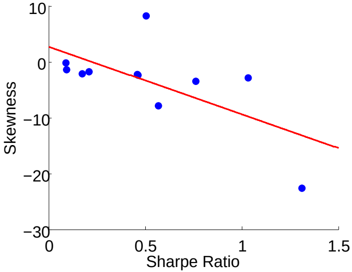

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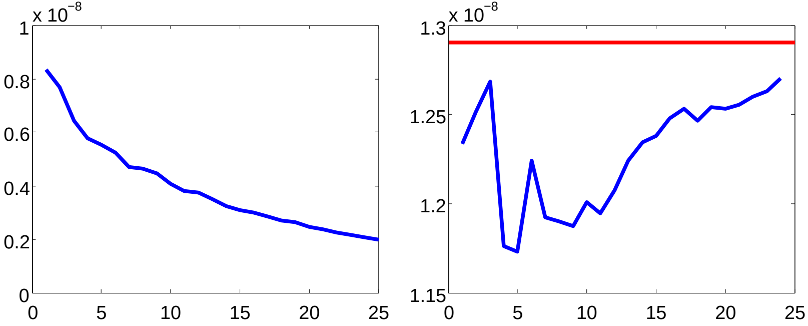
Now that we have captured the factors responsible for skewness and kurtosis, we can

- Do asset allocation: optimize with respect to the factors $w^\top := h^\top Q$ first
- Measure and understand the effect of weighting changes on skewness and kurtosis

Regularization and optimal number of factors



Reconstruction and generalization error $\times$ number of factors for a 50-stock portfolio.

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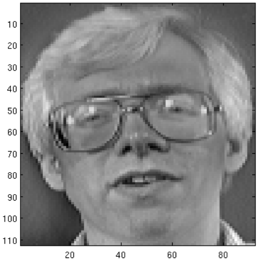

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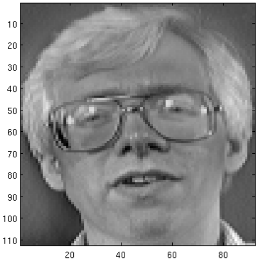

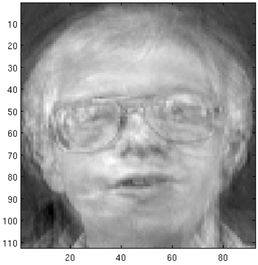

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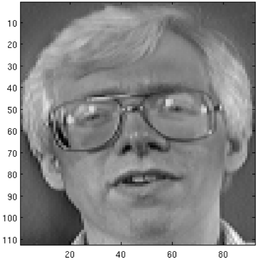

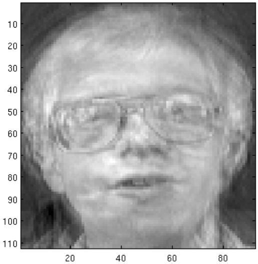

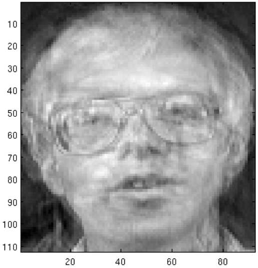

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- Estimated out-of sample higher-order portfolio statistics and optimized with them.
- Performed dimension reduction incorporating higher-order statistics.

A question

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- What is the distribution of the third and fourth cumulant tensors?

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End

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