Canonical Correlation Analysis for Functional Data and Stochastic Processes

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IMS New Researchers Conference, Baltimore, Maryland July 2009

August 2009

Presentation Outline

- Introduction
- Mathematical Preliminaries
- New Congruence Relation Results
- Literature Review of Canonical Correlation

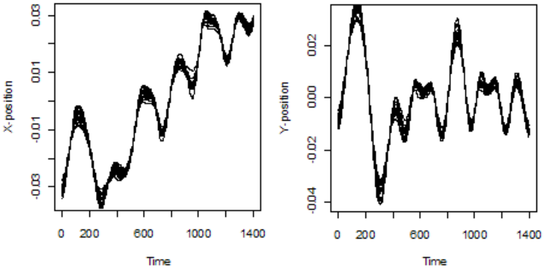


Figure: The X and Y positions of the tip of the pencil as "fda" is written.

- Observations consist of sample paths of stochastic processes on an index set E
- Sample paths are curves (e.g., $E = [a, b]$) or infinite sequences (e.g., $E = \mathbb{N}$).

Multivariate CCA vs Functional CCA

Multivariate CCA:

- Determine a pair of linear combinations $U_1 = \mathbf{a}_1'\mathbf{X}_1$ and $V_1 = \mathbf{b}_1'\mathbf{X}_2$ which have the largest correlation.
- Subsequent canonical correlations and variables $\{\mathbf{a}_j'\mathbf{X}_1, \mathbf{b}_j\mathbf{X}_2\}_{j=2}^{\min(N_1,N_2)}$ are obtained similarly subject to being uncorrelated with those that have already been determined.

Functional CCA:

- Covariance operators replaces covariance matrices.
- Pairs of "weight" functions $\{\alpha_j(\cdot), \beta_j(\cdot)\}_{j=1}^{\infty}$ are the analogues of weight vectors $\{\mathbf{a}_j, \mathbf{b}_j\}_{j=1}^{\min(N_1,N_2)}$.
- Intuitive solution to FCCA problem entails finding $\{U_j = \int \alpha_j X_1, V_j = \int \beta_j X_2\}_{j=1}^{\infty}$ which are maximally correlated with one another given that each pair is uncorrelated with previous pairs in the sequence.

Hilbert Spaces

$L^2(E)$

Let $(E, \mathcal{B}, \nu)$ denote a measure space, where E represents a subset of $\mathbb{R}$, $\mathcal{B}$ is the σ-algebra generated by the open sets of E and ν is a measure that is absolutely continuous with respect to the Lebesgue measure. The space $L^2(E)$ consists of all real-valued square integrable functions on E with the inner product $(f, g)_{L^2(E)} = \int_E fg d\nu$ for all $f, g \in L^2(E)$.

$L^2\{\Omega, \mathcal{A}, P\}$

Let $\{\Omega, \mathcal{A}, P\}$ be a probability space. The space of all second-order random variables on $\{\Omega, \mathcal{A}, P\}$ is a Hilbert space with inner product $(U, V)_{L^2\{\Omega, \mathcal{A}, P\}} := \mathrm{E}[UV]$.

The Hilbert Space Spanned by the Process

L^2_X Definition

Suppose that $\{X(t) : t \in E\}$ is a zero mean second order real-valued stochastic process on E . Let $\mathcal{E}_m = \{t_1, \ldots, t_m\} \subset E$ denote any finite collection of indices and let $\mathbf{X}_m = (X(t_1), \ldots, X(t_m))'$ with $\mathbf{K}_m = \mathrm{Var}(\mathbf{X}) = \{K(t_i, t_j)\}_{i,j=1}^m$. The Hilbert space spanned by the process X, denoted L^2_X, is the Cauchy completion of the set of all random variables having the form $U = \mathbf{a}_m'\mathbf{X}$, where $\mathbf{a}_m = (a_1, \ldots, a_m)' \in \mathbb{R}^m$ is subject to

$$\mathbf{a}_m \in \ker(\mathbf{K}_m)^{\perp}$$

for every collection of indices $\mathcal{E}_m \subset E$ and $m = 1, 2, \ldots$. The inner product between the two elements $U = \mathbf{a}_m'\mathbf{X}_m$ and $V = \mathbf{b}_n'\mathbf{X}_n$ in L^2_X is given by

$$(U, V)_{L^2_X} = \mathrm{Cov}[U, V] = \mathrm{Cov}[\mathbf{a}_m'\mathbf{X}_m, \mathbf{b}_n'\mathbf{X}_n] = \mathbf{a}_m'\mathbf{K}_{mn}\mathbf{b}_n.$$

Reproducing Kernel Hilbert Spaces (RKHS)

Definition of a reproducing kernel (r.k.)

Let $\mathcal{H}$ be a Hilbert space of functions on some set E. A bivariate function $K : E \times E \mapsto \mathbb{R}$ is said to be a reproducing kernel for $\mathcal{H}$ if

1. for every $t \in E$, $K(\cdot, t) \in \mathcal{H}$ and
2. for every $t \in E$ and $f \in \mathcal{H}$, $f(t) = (f, K(\cdot, t))_{\mathcal{H}}$.

When (i) and (ii) hold, $\mathcal{H}$ is said to be a reproducing kernel Hilbert space (RKHS) with reproducing kernel K and will be denoted by $\mathcal{H}(K)$. We will refer to (ii) as the reproducing property.

Theorem

A Hilbert space $\mathcal{H}$ possess a r.k. if and only if all evaluation functionals $\varepsilon_t(f) = f(t)$, $f \in \mathcal{H}$ for fixed $t \in E$ are continuous.

Relationship between Reproducing Kernels and Kernels for Integral Operators

Integral operators are Hilbert-Schmidt

Suppose E is a Lebesgue measurable subset of $\mathbb{R}$ and $K : E \times E \mapsto \mathbb{R}$ is a positive definite function. For each $s \in E$, suppose further that $K_s(\cdot) = K(s, \cdot)$ is dominated by a Lebesgue integrable function on E. Then the integral operator L defined by

$$(Lf)(s) = \int_E K(s,t)f(t)dt = (K(s,\cdot), f)_{L^2(E)},$$

is Hilbert-Schmidt.

- Every positive definite kernel K has an associated RKHS, but
- the $L^2(E)$ inner product is not capable of producing the reproducing property since $(Lf)(t) = (f, K(\cdot, t))_{L^2(E)} \neq (If)(t) = f(t)$.

The Importance of the RKHS Inner Product and Norm

$$K(s,t) = \sum_{i=1}^{\mathrm{rank}(L)} \lambda_i \phi_i(s) \phi_i(t).$$

$$K^{1/2}(s,t) \equiv \sum_{i=1}^{\mathrm{rank}(L)} \lambda_i^{1/2} \phi_i(s) \phi_i(t),$$

Nasheed and Wahba (1974)

The RKHS $\mathcal{H}(K)$ consists of functions of the form

$$f(\cdot) = \int_E g(s) K^{1/2}(\cdot, s) d\nu(s) = (L^{1/2} g)(\cdot),$$

for some $g \in \ker(L)^{\perp}$. The inner product in $\mathcal{H}(K)$ is $(f_1, f_2)_{\mathcal{H}(K)} = (g_1, g_2)_{L^2(E)}$ where $g_1, g_2 \in \ker(L)^{\perp}$ are the minimal $L^2(E)$ norm solutions of $f_i = L^{1/2} g_i$, $i = 1, 2$.

The Importance of the RKHS Inner Product and Norm

- Consider the problem of finding a solution to $L^{1/2}g = f$.
- It can be solved uniquely if and only if f satisfies the Picard criterion

$$\sum_{i=1}^{\mathrm{rank}(L)} \frac{(f,\phi_i)^2_{L^2(E)}}{\lambda_i} < \infty,$$

- Singular values decay faster than Fourier coefficients.
- So, $L^{1/2}g = f$ has a solution $\Leftrightarrow f \in \mathcal{H}(K)$ and for any $f \in \mathcal{H}(K)$ there corresponds a unique element $g_f \in L^2(E)$ of minimal norm given by $g_f := L^{1/2\dagger} f = L^{-1/2} f$
- Consequently, $\mathcal{H}(K) = \overline{(\mathrm{Im} L^{1/2})} = \ker(L)^{\perp}$ under the norm,

$$(f_1, f_2)_{\mathcal{H}(K)} = (L^{-1/2} f_1, L^{-1/2} f_2)_{L^2(E)}$$

The Importance of the RKHS Inner Product and Norm

Eubank and Hsing (2006)

The Hilbert spaces $\overline{(\mathrm{Im}L^{1/2})} = \ker(L)^{\perp}$ and $\mathcal{H}(K)$ are congruent under the mapping $\Gamma : \ker(L)^{\perp} \mapsto \mathcal{H}(K)$ defined by

$$\Gamma(g)(\cdot) := \int_E g(s)K^{1/2}(\cdot, s)d\nu(s) = \sum_{i=1}^{\mathrm{rank}(L)} \sqrt{\lambda_i} g_i \phi_i(\cdot),$$

where $g = \sum_{i=1}^{\mathrm{rank}(L)} (g, \phi_i)\phi_i = \sum_{i=1}^{\mathrm{rank}(L)} g_i\phi_i \in \ker(L)^{\perp}$. The inverse mapping

$$\Gamma^{-1}(f)(\cdot) := \sum_{i=1}^{\mathrm{rank}(L)} f_i\phi_i(\cdot).$$

for $f = \sum_{i=1}^{\mathrm{rank}(L)} \sqrt{\lambda_i} f_i \phi_i(\cdot) \in \mathcal{H}(K)$ is also the adjoint so that Γ is unitary.

Covariance Kernels and Operators of a Stochastic Process

Definition

Let $\{X(t), t \in E\}$ be a real, second order Hilbert space valued process. The covariance function of the process is a mapping from $E \times E \mapsto \mathbb{R}$ defined by

$$K(s,t) = \mathrm{Cov}(X(t), X(s)) = \mathrm{E}[X(s)X(t)].$$

The covariance operator S is defined by

$$(y, Sz)_{L^2(E)} \equiv \mathrm{Cov}((X,y)_{\mathcal{H}}, (X,z)_{\mathcal{H}})$$

and is an integral operator with the covariance function as its kernel

$$(Sg)(t) = \int_E K(s,t)g(s)d\nu(s) = (K(\cdot,t), g)_{L^2(E)}.$$

Karhunen Loeve Expansion

Karhunen–Loeve theorem

Let $Z_i \equiv (X, \phi_i)_{L^2(E)}$, with ϕ_i the eigenvector of S associated with λ_i, then

$$X(t) = \sum_{i=1}^{\mathrm{rank}(S)} Z_i \phi_i(t), \ t \in E,$$

where the convergence is in terms of mean and occurs pointwise on E.

- The $\{Z_n\}$ are orthogonal in the sense that
  $$\mathrm{Cov}[Z_m, Z_n] = \mathrm{E}[(X, \phi_m)_{\mathcal{H}}(X, \phi_n)_{\mathcal{H}}] \equiv (\phi_m, S\phi_n)_{\mathcal{H}} = \lambda_m \delta_{m,n}.$$
- If for $\lambda_n > 0$ if we define $\tilde{Z}_n \equiv Z_n/\sqrt{\lambda_n}$ to be the standardized orthogonal random variables, then

$$X(t) = \sum_{n=1}^{\mathrm{rank}(S)} \sqrt{\lambda_n} \tilde{Z}_n \phi_n(t), \ t \in E.$$

The Congruence from $\mathcal{H}(K)$ to L^2_X

Parzen-Loeve Isometry

The mapping $\Psi : \mathcal{H}(K) \mapsto L^2_X$ defined by the relation

$$\Psi\left(\sum_i a_i K(\cdot, t_i)\right) \mapsto \sum_i a_i X(t_i).$$

is congruent; i.e. Ψ is bijective and inner product preserving.

Proof.

$$\begin{aligned} \|\sum_i a_i X(t_i)\|^2_{L^2_X} &= \sum_{i,j} a_i a_j K(t_i, t_j) \\ &= \sum_{i,j} (a_i K(\cdot, t_i), a_j K(\cdot, t_j))_{\mathcal{H}(K)} \\ &= \|\sum_i a_i K(\cdot, t_i)\|^2_{\mathcal{H}(K)}. \end{aligned}$$

The Congruence from $\mathcal{H}(K)$ to L^2_X

Alternate Form of Parzen-Loeve Isometry

The RKHS $\mathcal{H}(K)$ consists of functions

$$f(\cdot) = \int_{\mathbb{N}} g(q)\phi(\cdot, q)d\nu(q) = \sum_{q=1}^{\mathrm{rank}(S)} \lambda_q g_q \phi_q(\cdot)$$

for some $g \in L^2(\mathbb{N})$ and the congruence mapping $\Psi : \mathcal{H}(K) \mapsto L^2_X$ is given by

$$\Psi(f) = \int_{\mathbb{N}} g(q)dZ(q) = \sum_{q=1}^{\mathrm{rank}(S)} g_q Z_q.$$

The RKHS inner product in this representation is

$$(f_1, f_2)_{\mathcal{H}(K)} = \sum^{\mathrm{rank}(S)} \lambda_j f_{1j} f_{2j}.$$

New Congruence Relation

Observe that the composition $\Omega := \Psi \circ \Gamma = \Psi\Gamma$ must also be a congruence.

The Congruence from $\ker(S)^{\perp}$ to L_X^2

The mapping $\Omega : \ker(S)^{\perp} \mapsto L_X^2$ given by $\Omega = \Psi\Gamma$ is congruent, unitary and has closed form

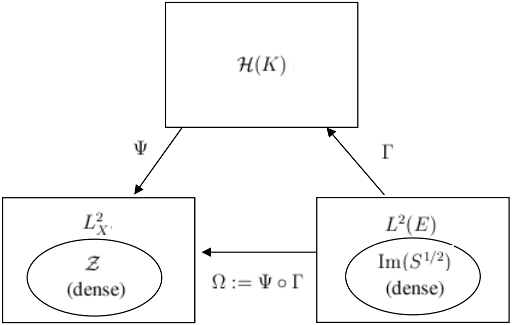
$$\Omega(f) = \begin{cases} (X, S^{-1/2}f)_{\mathcal{H}} \text{ whenever } f \in \mathrm{Im}(S^{1/2}) \text{ and,} \\ \sum_{i=1}^{\infty} f_i \tilde{Z}_i, \text{ for } f \in \left(\overline{\mathrm{Im}(S^{1/2})} \setminus \mathrm{Im}(S^{1/2})\right). \end{cases}$$

where $S^{-1/2}$ denotes the Moore-Penrose inverse of $S^{1/2}$. For any $U = \sum_{i=1}^{\mathrm{rank}(S)} f_i Z_i \in L_X^2$ the inverse mapping is given by

$$\Omega^{-1}(U) = S^{1/2} f,$$

with $f = \sum_{i=1}^{\mathrm{rank}(S)} f_i \phi_i \in \ker(S)^{\perp}$.

Hilbert Space Congruence Relations



$\mathcal{H}(K)$

Ψ

Γ

L^2_X

$\mathcal{Z}$ (dense)

$\Omega := \Psi \circ \Gamma$

$L^2(E)$

$\mathrm{Im}(S^{1/2})$ (dense)

Another New Result

An important subspace of L_X^2 is the subspace expressible by an $L^2(E)$ inner product: i.e.,

$$\mathcal{Z} := \{(X, f)_{L^2(E)} \mid f \in L^2(E)\} \subset L_X^2.$$

$\mathcal{Z}$ Theorem

The following relations hold

1. $\mathcal{Z} = \{(X, f)_{L^2(E)} \mid f \in \ker(S)^{\perp}\}$,
2. $\mathcal{Z} = \Omega(\mathrm{Im}(S^{1/2}))$,
3. $\overline{\mathcal{Z}} = L_X^2$, where $\overline{\mathcal{Z}}$ denotes the closure of L_X^2.

New Form of Parzen–Loéve Congruence Relation

Parzen–Loéve Isometry

For any $\tilde{f} = \sum_{i=1}^{\mathrm{rank}(S)} \lambda_i f_i \phi_i \in \mathcal{H}(K)$ and
$f = \Gamma^{-1}(\tilde{f}) = \sum_{i=1}^{\mathrm{rank}(S)} (\sqrt{\lambda}_i f_i)\phi_i \in \ker(S)^{\perp}$,

$$\Psi(\tilde{f}) = \begin{cases} (X, S^{\dagger}\tilde{f})_{\mathcal{H}}, & \text{whenever } \tilde{f} \in \Gamma(\mathrm{Im}(S^{1/2})) \text{ and,} \\ \sum_{i=1}^{\infty} (\sqrt{\lambda}_i f_i)\tilde{Z}_i, & \text{for } f \in \left(\overline{\mathrm{Im}(S^{1/2})} \setminus \mathrm{Im}(S^{1/2})\right), \end{cases}$$

with $S^{\dagger}$ denoting the Moore-Penrose inverse of S. The inverse mapping for any $U = (X, f)_{\mathcal{H}} = \sum_{i=1}^{\mathrm{rank}(S)} f_i (X, \phi_i)_{\mathcal{H}}$ with $f \in \ker(S)^{\perp}$ is

$$\Psi^{-1}(U) = Sf.$$

Tensor Form of Isometries

Note that

$$\{\tilde{Z}_i \equiv (X, \phi_i)_{\mathcal{H}}/\sqrt{\lambda_i} = \Omega(\phi_i)\}_{i=1}^{\mathrm{rank}(S)} \text{ and}$$

$$\{\tilde{\phi}_i \equiv \sqrt{\lambda_i}\phi_i = \Gamma\phi_i\}_{i=1}^{\mathrm{rank}(S)}$$

are complete orthonormal systems (CONS) for L_X^2 and $\mathcal{H}(K)$, respectively. Thus the tensor forms of the isometries are

$$\begin{aligned}
\Gamma &= \sum_{i=1}^{\mathrm{rank}(S)} \phi_i \otimes_{L^2(E)} \tilde{\phi}_i, & \Gamma^{-1} &= \sum_{i=1}^{\mathrm{rank}(S)} \tilde{\phi}_i \otimes_{\mathcal{H}(K)} \phi_i, \\
\Psi &= \sum_{i=1}^{\mathrm{rank}(S)} \tilde{\phi}_i \otimes_{\mathcal{H}(K)} \tilde{Z}_i, & \Psi^{-1} &= \sum_{i=1}^{\mathrm{rank}(S)} \tilde{Z}_i \otimes_{L_X^2} \tilde{\phi}_i, \text{ and,} \\
\Omega &= \sum_{i=1}^{\mathrm{rank}(S)} \phi_i \otimes_{L^2(E)} \tilde{Z}_i, & \Omega^{-1} &= \sum_{i=1}^{\mathrm{rank}(S)} \tilde{Z}_i \otimes_{L_X^2} \phi_i.
\end{aligned}$$

A New Result Which Derives from Developments in the "French School'

- For $i = 1, 2$, $L_i : \mathcal{H}_i \mapsto L^2_{X_i}$ be given by

$$L_i f \equiv (f, X_i)_{\mathcal{H}_i}.$$

- The adjoint of L_i is the mapping $L_i^* : L^2_{X_i} \mapsto \mathcal{H}_i$ that has the form

$$L_i^*(U) \equiv \mathrm{E}[UX_i].$$

- For $i, j = 1, 2$ the operators L_i and L_j satisfy

$$(L_i f, L_j g)_{L^2_X} = \mathrm{E}[(f, X_i)_{\mathcal{H}_i}(g, X_j)_{\mathcal{H}_j}] = (f, S_{ij} g)_{\mathcal{H}_i}, \qquad (1)$$

thus $S_{ij} = L_i^* L_j$, for $i, j = 1, 2$.

The French School Approach

Despite the finite dimensional restriction there are many powerful results which can be obtained from the French school. For example, the following apparently new result

Extension of Result from Khatari

$$S_1 S_1^\dagger S_{12} = S_{12} \quad \text{and} \quad S_2 S_2^\dagger S_{21} = S_{21}.$$

Proof. Notice that

$$S_1 S_1^\dagger S_{12} = (L_1^* L_1)(L_1^* L_1)^\dagger (L_1^* L_2) = L_1^* L_2 = S_{12},$$

is true in both finite and infinite dimensions. ◊

The He, Muller and Wang Approach to CCA

He, Muller and Wang (200,2002) (Henceforth HMW) define the first squared canonical correlation ρ_1^2 and associated weight functions or vectors u_1 and v_1 by

$$\begin{aligned} \rho_1^2 &= \sup_{\substack{u \in L^2(E_1) \\ v \in L^2(E_2)}} \mathrm{Cov}^2[(u, X_1)_{L^2(E_1)}, (v, X_2)_{L^2(E_2)}] \\ &= \mathrm{Cov}^2[(u_1, X_1)_{L^2(E_1)}, (v_1, X_2)_{L^2(E_2)}], \end{aligned} \tag{2}$$

where u and v are subject to

$$\mathrm{Var}[(u, X_1)_{L^2(E_1)}] = \mathrm{Var}[(v, X_2)_{L^2(E_2)}] = 1. \tag{3}$$

The He, Muller and Wang Approach to CCA

- Optimization is equivalent to the cross-correlation operator R defined by
$$R = S_1^{-1/2} S_{12} S_2^{-1/2}. \tag{4}$$
- $\Rightarrow$ find eigenvalues for
$$RR^* = S_1^{-1/2} S_{12} S_2^{\dagger} S_{21} S_1^{-1/2}$$
- Square roots of covariance operators of infinite dimensional Hilbert space valued processes are not invertible.
- Solution: restrict to set set of functions $S_1^{1/2}(\ker(S)^{\perp})$

The Eubank and Hsing Approach

- EH define the first squared canonical correlation ρ_1^2 and the associated canonical variables $U_1 \in L^2_{X_1}$ and $V_1 \in L^2_{X_2}$ by

$$\rho_1^2 = \sup_{U \in L^2_{X_1}, V \in L^2_{X_2}} \mathrm{Cov}^2[U, V] = \mathrm{Cov}^2[U_1, V_1], \tag{5}$$

where U and V are subject to

$$\mathrm{Var}[U] = \mathrm{Var}[V] = 1. \tag{6}$$

- Utilizing Ψ_i EH re-formulate

$$\rho_1^2 = \sup_{\tilde{f} \in \mathcal{H}(K_1), \tilde{g} \in \mathcal{H}(K_2)} \mathrm{Cov}^2[\Psi_1(\tilde{f}), \Psi_2(\tilde{g})] = \mathrm{Cov}^2[\Psi_1(\tilde{f}_1), \Psi_2(\tilde{g}_1)], \tag{7}$$

The Eubank and Hsing Approach

- This optimization is equivalent to

$$\begin{aligned}\mathrm{Cov}[\Psi_1(\tilde{f}),\Psi_2(\tilde{g})] &= (\tilde{f}(\cdot),(K_{12}(\cdot,*),\tilde{g}(*))_{\mathcal{H}(K_2)})_{\mathcal{H}(K_1)}\\ &:= (\tilde{f},T\tilde{g})_{\mathcal{H}(K_1)}\end{aligned}$$

  with the mapping $T:\mathcal{H}(K_2)\mapsto\mathcal{H}(K_1)$ given by

$$(T\tilde{g})(s)=(K_{12}(s,\cdot),\tilde{g}(\cdot))_{\mathcal{H}(K_2)}. \tag{8}$$

- Perform singular value decomposition of T

$$T=\sum_{k=1}^{\mathrm{rank}(T)}\rho_k\tilde{g}_k\otimes_{\mathcal{H}(K_2)}\tilde{f}_k.$$

Comparison of CCA Approaches

- Let $\gamma_{ij} = \mathrm{E}[Z_i Z_j]$ and $\rho_{jk} \equiv \frac{\gamma_{jk}}{\sqrt{\lambda_{1j}\lambda_{2k}}}$.
- Then

$$\begin{aligned}
(T\tilde{g})(s) &= (K_{12}(s,\cdot), \tilde{g}(\cdot))_{\mathcal{H}(K_2)} \\
&= \sum_{j=1}^{\mathrm{rank}(S_1)} \sum_{k=1}^{\mathrm{rank}(S_2)} \gamma_{jk}\phi_{1j}(s)(\phi_{2k}, \tilde{g})_{\mathcal{H}(K_2)} \\
&= \sum_{j=1}^{\mathrm{rank}(S_1)} \sum_{k=1}^{\mathrm{rank}(S_2)} (\rho_{jk}\sqrt{\lambda_{1j}\lambda_{2k}})\phi_{1j}(s)(\phi_{2k}, \tilde{g})_{\mathcal{H}(K_2)}.
\end{aligned}$$

Comparison of CCA Approaches without Regularization

$$\begin{aligned}
T &= \sum_{j=1}^{\mathrm{rank}(S_1)} \sum_{k=1}^{\mathrm{rank}(S_2)} \rho_{jk} \tilde{\phi}_{2k} \otimes_{\mathcal{H}(K_2)} \tilde{\phi}_{1j} \\
&= \sum_{j=1}^{\mathrm{rank}(S_1)} \sum_{k=1}^{\mathrm{rank}(S_2)} \rho_{jk} [(\Gamma_2 \phi_{2k}) \otimes_{\mathcal{H}(K_2)} (\Gamma_1 \phi_{1j})] \\
&= \sum_{j=1}^{\mathrm{rank}(S_1)} \sum_{k=1}^{\mathrm{rank}(S_2)} \rho_{jk} [(\phi_{2k} \Gamma_2^{-1}) \otimes_{L^2(E_2)} (\Gamma_1 \phi_{1j})] \\
&= \Gamma_1 \left[\sum_{j=1}^{\mathrm{rank}(S_1)} \sum_{k=1}^{\mathrm{rank}(S_2)} \rho_{jk} [\phi_{2k} \otimes_{L^2(E_2)} \phi_{1j}] \right] \Gamma_2^{-1} \\
&= \Gamma_1 R \Gamma_2^{-1} = \Gamma_1 S_1^{-1/2} S_{12} S_2^{-1/2} \Gamma_2^{-1}.
\end{aligned}$$

Comparison of CCA Approaches without Regularization

- Since Γ_1 and Γ_2 are unitary, T is unitarily equivalent to R and the two methods agree when both are defined.
- Because Γ_2 is a bijection, the domain of $\Gamma_1^{-1} T \Gamma_2$ is $\ker(S_2)^\perp$.
- On the other hand, the domain of R must be restricted to $F_2 = S_2^{1/2}\left(\ker(S_2)^\perp\right)$, which is a dense proper subset of $\ker(S_2)^\perp$.
- RKHS approach has solutions on the boundary $\left(\overline{\mathrm{Im}(S_2^{1/2})} \setminus \mathrm{Im}(S_2^{1/2})\right)$, whereas $L^2(E)$ based approaches do not.

Comparison of CCA Approaches without Regularization

- When $\{f, g\} := \{S_1^{1/2}u, S_2^{1/2}v\}$ range over $\{\mathrm{Im}(S_1^{1/2}), \mathrm{Im}(S_2^{1/2})\}$ the corresponding random variables $U = \Omega_1(f) = (X_1, u)_{L^2(E_1)}$ and $V = \Omega_2(g) = (X_2, v)_{L^2(E_2)}$ range over $\{\mathcal{Z}_1, \mathcal{Z}_2\}$
- Another perspective, in HMW approach

$$\begin{aligned} \rho_1^2 &= \sup_{\substack{f \in \mathrm{Im}\left(S_1^{1/2}\right) \\ g \in \mathrm{Im}\left(S_2^{1/2}\right)}} \mathrm{Cov}^2[(f, X_1)_{L^2(E_1)}, (g, X_2)_{L^2(E_2)}], \\ &= \sup_{\substack{U \in \mathcal{Z}_1 \\ V \in \mathcal{Z}_2}} \mathrm{Cov}^2[U, V]. \end{aligned}$$

- In the EH approach

$$\rho_1^2 = \sup_{\substack{U \in L^2_{X_1} \\ V \in L^2_{X_2}}} \mathrm{Cov}^2[U, V].$$

Thank you Tracy and Bala and all of YOU for listening!!

Thank you for listening!!