De Wijs Process and Modeling Disease Risk

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Outline for the talk

- **overview**
- Two approaches to disease mapping
  - **Gaussian Markov random fields**
  - **Geostatistical**
- Combining the above two approaches via **de Wijs Process**
- End with an example on **disease mapping**

Overview

Interest on **maps of risks** of non-infectious diseases over a **geographic region**.

These maps

- reveal heterogeneity
- indicate areas of high relative risk that may require screening and intervention
- help detect effects of socio-economic conditions and exposure to environmental hazards
- provide clues to the etiology of diseases
- use counts aggregated over zip-codes, counties or states to protect individual identities.

Examples: **Clayton and Kaldor (1987), Besag et al. (1991), Best et al. (2001), Kelsall and Wakefield (2002), Banerjee et al. (2004) and Rue and Held (2005)**....

Two statistical approaches are **Markov random fields** and **geostatistics**.

Example of irregular regions

Markov random field approach (Besag et al., 1991, Rue and Held, 2005)

Consider the **conditional Poisson model**

$$Y(A_i) \sim \text{independent Poisson}\Big(c(A_i)R(A_i)\Big),\ i = 1, \ldots, n, \tag{1}$$

with

$$\log R(A_i) = T(A_i)\beta + U(A_i) + \epsilon(A_i),$$

	$y(A_i)$	=	**observed number** of cases of disease in region (e.g., county) A_i
	$c(A_i)$	=	expected number of cases of disease in region A_i
	$R(A_i)$	=	**log relative risk** in region A_i
where	$T(A_i)$	=	area-specific covariate,
	β	=	covariate effect
	$\epsilon(A_i)$	=	**unstructured component** or residual
	$U(A_i)$	=	A Markov random field

Assumes two areas are **adjacent** or **neighbors** when they share a common boundary.

Conditionally, $U(A_i)$ follows $N\Big(\sum_{j\sim i} U(A_j)/l_i, \sigma_U^2/l_i\Big)$.

$j \sim i$ implies that area A_j is adjacent to A_i and l_i is number of neighbors of A_i.

Geostatistical approach (Kelsall and Wakefield (2002)

Consider the **conditional Poisson model** driven by a Cox process

$$Y(A_i) \sim \text{independent Poisson}\Big(\sum_k \int_{A_i} c_k(s) R_k(s)\, ds\Big), \; i = 1, \ldots, n. \tag{2}$$

with

$$R_k(s) = R(s), \quad R(A_i) = \int_{A_i} f_i(s) e^{T(s)\beta + U(s) + \epsilon(s)}\, ds, \tag{3}$$

where

k	$=$	population stratum, $R_k(s)$ log relative risk in stratum k, location s
$f_i(s)$	$=$	weighted average of the stratum-specific population density
$c(A_i)$	$=$	$\sum_k \int_{A_i} c_k(s)\, ds$ is expected number of cases of disease in region A_i
$T(s)$	$=$	location-wise covariate information
β	$=$	covariate effect
$\epsilon(s)$	$=$	Gaussian white noise process
$U(s)$	$=$	A continuum spatial process

Includes **area** and **population information**, but computationally challenging.

Combining Markov random fields with geostatistical approach

Let $\{X_{u,v} : (u,v) \in \mathcal{Z}^2\}$ be Gaussian with **conditional** means and variances

$$\mathrm{E}\,(X_{u,v}|\ldots) = \gamma(x_{u-1,v} + x_{u+1,v} + x_{u,v-1} + x_{u,v+1}), \quad \mathrm{var}\,(X_{u,v}|\ldots) = \sigma_X^2, \qquad (4)$$

(i) $2\gamma < \frac{1}{2} \;\Rightarrow$ stationary autoregression e.g. Lévy (1948), Besag (1974).

(ii) $2\gamma = \frac{1}{2} \;\Rightarrow$ **intrinsic** case: Künsch (1987), Besag & Kooperberg (1995).

Limit diagram for stationary and intrinsic case

$2\gamma \to \frac{1}{2}$

Stationary autoregression ——————> intrinsic autoregression

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Bessel_0 process ——————————-> De Wijs process

Idea is to use exponential functional of the de Wijs process.

In other words, use $e^{X_{u,v}}$ to **approximate** $U(s)$!!!

Combining Markov random fields with geostatistical approach

Embed study region in a fine mesh

$$\mathcal{L} = \{(u, v) : u = 1, \dots I, v = 1, \dots, J\}, \tag{5}$$

Each grid point (u, v) is associated with a regular cell $D(u, v)$. Use

$$A_i = \cup_{(u,v)\in A_i} D(u, v). \tag{6}$$

Let $m(A_i)$ be equal to the number of cells $D(u, v)$ in A_i. Consider

$$f_i(u, v) = \int_{D(u,v)} f_i(s), \, ds.$$

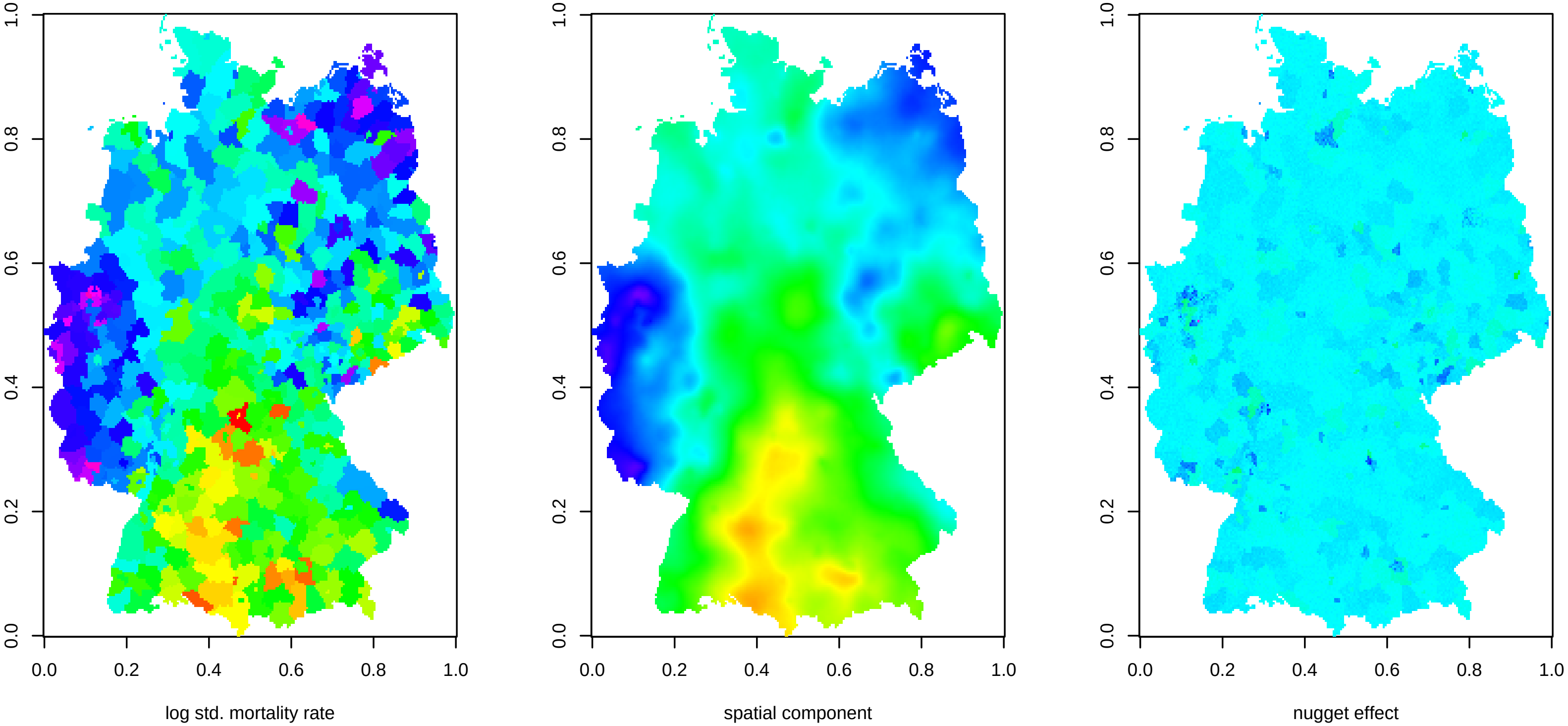
Finally, use

$$R(A_i) = \sum_{(u,v)\in A_i} f_i(u, v) e^{T(u,v)\beta + X_{u,v} + \epsilon_{u,v}}. \tag{7}$$

The **continuum random field theory** for another day!

This theory relies on **generalized Gaussian process**, **subordinate random fields** and the famous **Dobrushin's expansion**.

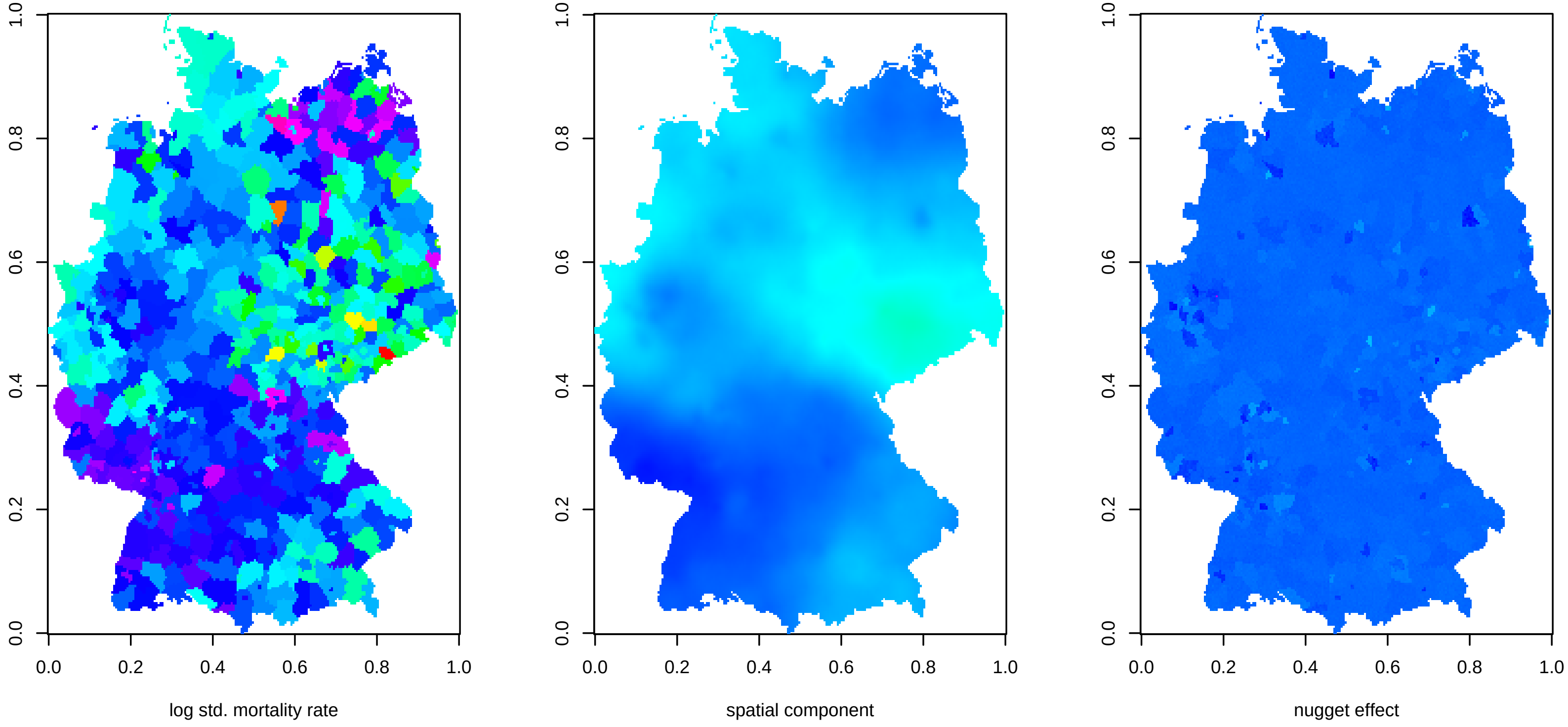
Lung cancer in Germany



Data in the period 1986-1990. 544 districts embeded in 289×214 girds.

See also Rue and Held, 2005.

Oral cancer in Germany



Data during the years 1986-1990. 544 districts embeded in 289 $\times$ 214 girds.

Joint disease mapping can also be done!

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