Chapter 11. Network Community Detection

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PubH 7475/8475

Outline

- Introduction
- Spectral clustering
- Hierachical clustering
- Modularity-based methods
- Model-based methods
- Key refs:
  1.Newman MEJ
  2. Zhao Y, Levina E, Zhu J (2012, Ann Statist 40:2266-2292).
  3. Fortunato S (2010, Physics Reports 486:75-174).
- R package `igraph`: drawing networks, calculating some network statistics, some community detection algorithms, ...

Introduction

- Given a binary (undirected) network/graph: $G = (V, E)$, $V = \{1, 2, ..., n\}$, set of nodes; E, set of edges. Adjacency matrix $A = (A_{ij})$: $A_{ij} = 1$ if there is an edge/link b/w nodes i and j; $A_{ij} = 0$ o/w. $(A_{ii} = 0)$
- Goal: assign the nodes into K "homogeneous" groups. often means dense connections within groups, but sparse b/w groups.
- Why? Figs 1-4 in Fortunato (2010).

Spectral clustering

- Laplacian $L = D - A$, or ...
  $D = \text{Diag}(D_{11}, ..., D_{nn})$, $D_{ii} = \sum_j A_{ij}$.
- Intuition:
  If a network separates perfectly into K communities, then L (or A) is block diagonal (after some re-ordering of the rows/columns).
  If not perfectly but nearly, then the eigenvectors of L are (nearly) linear combinations of the indicator vectors.
- Apply K-means (or ..) to a few (K) eigenvectors corresponding to the smallest eigenvalues of L.
  (Note: the smallest eigen value is 0, corresponding to eigenvector 1.)
- Widely used; some theory (e.g consistency).

Modified spectral clustering

- SC may not work well for sparse networks.
- Regularized SC (Qin and Rohe): replace D with $D_\tau = D + \tau I$ for a small $\tau > 0$.
- SC with perturbations (Amini, Chen, Bickel, Levina, 2013, Ann Statist 41: 2097-2122):
  regularize A by adding a small positive number on a random subset of off-diagonals of A.

Hierarchical clustering

- Need to define some similarity or distance b/w nodes.
- Euclidean distance: $A_{i.} = (A_{i1}, A_{I2}, ..., A_{in})'$,

$$x_{ij} = ||A_{i.} - A_{j.}||_2$$

- Or, Pearson's corr,

$$x_{ij} = \text{corr}(A_{i.}, A_{j.})$$

- Then apply a hierarchical clustering.
  can be used to re-arrange the rows/columns of A to get a nearly block-diagonal A.
- Fig 3 in Neuman.
- Fig 2 in Meunier et al (2010).

Algorithms based on edge removal

- Divisive: edges are progressively removed.
- Which edges? "bottleneck" ones.
- *edge betweenness* is defined to be the number of shortest paths between all pairs of all nodes that run through the two nodes.
- Algorithm (Girvam and Neuman 2002, PNAS):
  1) calculate *edge betweenness* for each remaining edge in a network;
  2) remove the edge with the higest *edge betweenness*;
  3) repeat the above until ...
- A possible stopping critarion: *modularity*, to be discussed.
- Fig 4 in Neuman.
- Remarks: slow; some modifications, e.g. a Monte Carlo version in calculating *edge betweenness* using only a random subset of all pairs; or use a different criterion.

Modularity-based methods

- Notation:
  degree of node i: $d_i = D_{ii} = \sum_{j=1}^{n} A_{ij}$,
  (twice) total number of edges: $m = \sum_{i=1}^{n} d_i$,
  Community assignment: $C = (C_1, C_2, ..., C_n)$; **unknown**,
  $C_i \in \{1, 2, ..., K\}$: community containing node i.
- Modularity:

$$Q = Q(C) = \frac{1}{2m} \sum_{i,j} \left(A_{ij} - \frac{d_i d_j}{m} \right) I(C_i = C_j).$$

- Intuition: obs'ed - exp'ed
- Goal: $\hat{C} = \arg\max_C Q(C)$
  Assumption: good to maximize Q, but ...
- Key: a **combinatorial** optimization problem!
  seeking exact solution will be too slow $\Longrightarrow$ many *approximate* algorithms, such as greedy searches (e.g. genetic algorithms, simulated annealing), relaxed algorithms, ...

- Very nonparametric?!
- Problems: resolution limit; too many local solutions.
  cannot detect small communities; why? an implicit null model.

Model-based methods

- Stochastic block model SBM (Holland et al 1983):
  1) a $K \times K$ probability matrix P;
  2) $A_{ij} \sim \text{Bin}(1, P_{C_i,C_j})$ independently.
- Simple; can model dense/weak within-/between-community edges.
  But, treat all nodes/edges in a community equally; cannot model *hub* nodes!
  Scale-free network: node degree distribution $Pr(k)$ is heavy-tailed; a power law.
- SBM with $K = 1$: Erdos-Renyi Random Graph.
- Degree-corrected SBM (DCSBM) (Karrer and Newman 2011):
  1) P; each node i has a degree parameter θ_i (with some constraints for identifiability);
  2) $A_{ij} \sim \text{Bin}(1, \theta_i \theta_j P_{C_i,C_j})$ independently

- More notations:
  $n_k(C) = \sum_{i=1}^n I(C_i = k)$, number of nodes in community k;
  $O_{kl} = \sum_{i,j=1}^n A_{ij} I(C_i = k, C_j = l)$, number of edges b/w communities $k \neq l$;
  $O_{kk} = \sum_{i,j=1}^n A_{ij} I(C_i = k, C_j = k)$, (twice) number of edges within community k;
  $O_k = \sum_{l=1}^K O_{kl}$, sum of node degrees in community k;
  $m = \sum_{i=1}^n d_i$, (twice) the number fo edges in the network.
- Objective function: A profile likelihood (profiling out nuisance parameters P and θ's based on a Poisson approximation to a binomial).
  Given a likelihood $L(C, P)$,
  a profile likelihood $L^*(C) = \max_P L(C, P) = L(C, \hat{P}(C))$.

- SBM:

$$Q_{SB}(C) = \sum_{k,l=1}^{K} (O_{kl} \log \frac{O_{kl}}{n_k n_l}).$$

- DCSBM:

$$Q_{DC}(C) = \sum_{k,l=1}^{K} (O_{kl} \log \frac{O_{kl}}{O_k O_l}).$$

- Neuman-Girvan modularity:

$$Q_{NG}(C) = \frac{1}{2m} \sum_{k} (O_{kk} - \frac{O_k^2}{m}).$$

- Remarks: Still a combinatorial optimization problem; better theoretical properties.
- Numerical examples in Zhao et al (2012).

Other topics

- Summary statistics for networks; e.g. clustering coeficient,...
- Weighted networks; with or without negative weights (e.g. Pearson's correlations).
- Overlapping communities.
- Time-varying (dynamic) networks.
- With covariates. How to model covariates?
- Fast (approximate) algorithms; theory.