Bayesian Sample Size Computations

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April 20, 2008
Suppose we want to take a sample \( y_1, \ldots, y_n \) iid \( \sim N(\theta, \sigma^2) \). So \( \bar{y} \sim N(\theta, \sigma^2/n) \). Assume \( \sigma^2 \) is known. \( n \) is not known.

Consider the classical hypothesis testing problem:
\( H_0 : \theta = \theta_0 \) against the alternative \( H_1 : \theta = \theta_1 > \theta_0 \).

Decision rule: Reject \( H_0 \) if \( \bar{y} > \theta_0 + \frac{\sigma}{\sqrt{n}} z_{1-\alpha} \), where \( \Phi(z_\alpha) = \alpha \) and \( \Phi(\cdot) \) is the standard normal cdf.
Review of classical sample size calculations

- Requiring the procedure to have a power of at least $1 - \beta$, we have:

$$1 - \beta \leq P(\text{Reject } H_0 \mid H_1) = P\left( \bar{y} > \theta_0 + \frac{\sigma}{\sqrt{n}} z_{1-\alpha} \mid \theta = \theta_1 \right)$$

$$= P\left( \frac{\sqrt{n}}{\sigma} (\bar{y} - \theta_1) > \frac{\sqrt{n}}{\sigma} (\theta_0 - \theta_1) + z_{1-\alpha} \right)$$

$$= P\left( Z > -\sqrt{n} \frac{\Delta}{\sigma} + z_{1-\alpha} \right)$$

$$= 1 - \Phi\left( -\sqrt{n} \frac{\Delta}{\sigma} + z_{1-\alpha} \right)$$

$$= \Phi\left( \sqrt{n} \frac{\Delta}{\sigma} - z_{1-\alpha} \right) = \Phi\left( \sqrt{n} \frac{\Delta}{\sigma} + z_{1-\alpha} \right),$$

where $\Delta = \theta_1 - \theta_0$ and, in the last steps, we have used the facts that $\Phi(-x) = 1 - \Phi(x)$ and $z_{1-\alpha} = -z_{\alpha}$. 
The preceding computations lead to

\[ \sqrt{n} \frac{\Delta}{\sigma} + z_\alpha \geq z_{1-\beta} \]

\[ \implies \sqrt{n} \frac{\Delta}{\sigma} \geq (z_{1-\beta} - z_\alpha) = -(z_\alpha + z_\beta) \]

\[ \implies n \geq (z_\alpha + z_\beta)^2 \left( \frac{\sigma}{\Delta} \right)^2 . \]

Thus, we arrive at the ubiquitous sample size formula:

The sample size formula:

\[ n = (z_\alpha + z_\beta)^2 \left( \frac{\sigma}{\Delta} \right)^2 . \]

For two-sided alternatives, one simply replaces \( \alpha \) by \( \alpha/2 \) in the above expression.
Suppose, we take the prior $\theta \sim N(\theta_1, \tau^2)$. We let $\tau^2 = \frac{\sigma^2}{n_0}$, where $n_0$ (prior sample size) reflects the precision of the prior relative to the data. This simplifies the calculations:

$$N \left( \theta \mid \theta_1, \frac{\sigma^2}{n_0} \right) \times N \left( \bar{y} \mid \theta, \frac{\sigma^2}{n} \right)$$

$$= N \left( \theta \mid \frac{n_0}{n + n_0} \theta_1 + \frac{n}{n + n_0} \bar{y}, \frac{\sigma^2}{n + n_0} \right).$$

Let $A_\alpha(\theta_0, \theta_1) = \{\bar{y} : P(\theta < \theta_0 \mid \bar{y}) < \alpha\}$. Note:

$$P(\theta < \theta_0 \mid \bar{y}) =$$

$$P \left\{ \frac{\sqrt{n + n_0}}{\sigma} \left( \theta - \frac{n_0 \theta_1 + n \bar{y}}{n + n_0} \right) < \frac{\sqrt{n + n_0}}{\sigma} \left( \theta_0 - \frac{n_0 \theta_1 + n \bar{y}}{n + n_0} \right) \right\}$$

$$= \Phi \left[ \frac{\sqrt{n + n_0}}{\sigma} \left( \theta_0 - \frac{n_0 \theta_1 + n \bar{y}}{n + n_0} \right) \right]$$
Thus,

\[ A_\alpha(\theta_0, \theta_1) = \left\{ \bar{y} : \frac{\sqrt{n + n_0}}{\sigma} \left( \theta_0 - \frac{n_0 \theta_1 + n \bar{y}}{n + n_0} \right) < z_\alpha \right\} \]

\[ = \left\{ \bar{y} : \theta_0 - \frac{n_0 \theta_1 + n \bar{y}}{n + n_0} < \frac{\sigma}{\sqrt{n + n_0}} z_\alpha \right\} \]

\[ = \left\{ \bar{y} : \theta_0 - \frac{n_0}{n + n_0} \theta_1 - \frac{n}{n + n_0} \bar{y} < \frac{\sigma}{\sqrt{n + n_0}} z_\alpha \right\} \]

\[ = \left\{ \bar{y} : \bar{y} > \theta_0 - \frac{n_0}{n} (\theta_1 - \theta_0) - \sqrt{\left(1 + \frac{n_0}{n}\right)} \frac{\sigma}{\sqrt{n}} z_\alpha \right\} \]

Note that as \( n_0 \to 0 \) (i.e. the prior becomes vague) \( A_\alpha(\theta_0, \theta_1) \) becomes identical to the critical region from classical hypothesis testing.
The Bayesian power or Bayesian assurance $\delta$ is defined as:

$$
\delta = P_{\bar{y}}(A_\alpha(\theta_0, \theta_1)) \\
= P_{\bar{y}}\{\bar{y} : P(\theta < \theta_0 | \bar{y}) < \alpha\} \\
= P_{\bar{y}}\{\bar{y} : P(\theta > \theta_0 | \bar{y}) > 1 - \alpha\} \\
= P_{\bar{y}}\left\{\bar{y} > \theta_0 - \frac{n_0}{n} (\theta_1 - \theta_0) - \sqrt{\left(1 + \frac{n_0}{n}\right) \frac{\sigma}{\sqrt{n}}} z_\alpha\right\}
$$
Bayesian sample size calculations

- The marginal distribution of $\bar{y}$ is given by:

$$\int N\left( \theta \mid \theta_1, \frac{\sigma^2}{n_0} \right) \times N\left( \bar{y} \mid \theta, \frac{\sigma^2}{n} \right) d\theta = N\left( \bar{y} \mid \theta_1, \frac{\sigma^2}{n + n_0} \right).$$

- Therefore, the Bayesian power or assurance is:

$$P_{\bar{y}} \left\{ \bar{y} > \theta_0 - \frac{n_0}{n} (\theta_1 - \theta_0) - \frac{\sqrt{n + n_0}}{n} \sigma z_\alpha \right\}$$

$$= P_{\bar{y}} \left\{ \bar{y} - \theta_1 > - \left(1 + \frac{n_0}{n}\right) (\theta_1 - \theta_0) - \frac{\sqrt{n + n_0}}{n} \sigma z_\alpha \right\}$$

$$= P \left( Z > -\sqrt{n + n_0} \left(1 + \frac{n_0}{n}\right) \frac{\theta_1 - \theta_0}{\sigma} - \left(1 + \frac{n_0}{n}\right) z_\alpha \right)$$

$$= \Phi \left( \sqrt{n + n_0} \left(1 + \frac{n_0}{n}\right) \frac{\theta_1 - \theta_0}{\sigma} + \left(1 + \frac{n_0}{n}\right) z_\alpha \right)$$
Rewriting the Bayesian power in terms of the relative precision \( \frac{n_0}{n} \) and \( n \), we obtain:

\[
\delta = \Phi \left( \sqrt{n} \left( 1 + \frac{n_0}{n} \right)^{3/2} \frac{\Delta}{\sigma} + \left( 1 + \frac{n_0}{n} \right) z_\alpha \right),
\]

where \( \Delta = \theta_1 - \theta_0 \). This is often called the \textit{critical difference} that needs to be detected.

This leads to:

\[
\delta(\Delta, n) = \Phi \left( \sqrt{n} \left( 1 + \frac{n_0}{n} \right)^{3/2} \left( \frac{\Delta}{\sigma} \right) + \left( 1 + \frac{n_0}{n} \right) z_\alpha \right),
\]

Note: For study design purposes, \( \sigma^2 \) and \( n_0 \) are assumed known and the Bayesian power curve is investigated as a function of \( \Delta \) and \( n \).
Given $n_0$, the Bayesian will compute the sample size needed to detect a critical difference of $\Delta$ with probability $1 - \beta$ as

$$n = \arg \min \{ n : \delta(\Delta, n) \geq 1 - \beta \}$$

As the prior becomes vague, $n_0 \to 0$ and:

$$\lim_{n_0 \to 0} \delta(\Delta, n) = \Phi \left( \sqrt{n} \frac{\Delta}{\sigma} + z_{\alpha} \right)$$

which is exactly the classical power curve. Now the Bayesian sample size formula coincides with the classical sample size formula:

Bayesian sample size with vague prior information:

$$n = (z_{\alpha} + z_{\beta})^2 \left( \frac{\sigma}{\Delta} \right)^2.$$